UNIVERSITY OF ALBERTA

LEARNING ALGEBRA PERSONALLY

BY

RALPH T. MASON ©

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy.

DEPARTMENT OF SECONDARY EDUCATION

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ABSTRACT

Twenty-two grade nine students learned algebra through a series of interactive small-group inquiry activities. The activities reflected a constructivist orientation to learning and teaching in their design and implementation. The purpose of this study was to illuminate the factors that affected students' success in interactive small-group inquiry learning and to illuminate how those factors were in turn affected by the students' learning processes. Data included classroom and small-group transcripts, student mathematics products, and student writing; this last included student "homework," optional daily writing by the student about the student's mathematical learning that day. To maintain the personal and interpersonal nature of the context and the study, the goals, theory, data, data interpretations, and conclusions are presented as a research story, an interpretive retelling of the events of the teaching and learning.

Students can learn how to learn mathematics well through interactive small-group inquiry, and this learning can be observed within the same frame which constructivist epistemology provides for content-oriented learning. Students repositioned themselves personally in terms of their relative autonomy and the teacher's authority with regard to the mathematical content, the learning processes, and what counts as student success. When learning to learn well in ways which are different from their prior experiences learning mathematics, students reoriented themselves epistemologically in regard to their sense of what counts as knowledge and understanding and learning. Students can also learn ontologically by coming to see themselves as students and learners in more complex ways compatible with the learning opportunities provided. Finally, students came to perceive the context in which they operated in more complex ways, as they interpreted the institutional aspects of their relationships with mathematics, with teachers, and with tests and marks.

All of this learning was personal, dependent on and building from the experiences and perspectives of each learner. This learning was interpersonal as well, dependent on and building from the relationships each learner developed with other learners and with the teacher. The teacher was a part of these personal and interpersonal processes in authentic ways, and the learners were subject to the teacher's influence to the extent that the teacher and the teaching were subject to the influence of the learners within the relationships which developed.

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Chapter 1. A Quiet Learner Speaks

Three grade nine students are about to begin a small-group inquiry activity. Benazhir, Rose, and Lorna are used to having a tape recorder on their desks, because they have been a part of a research project for five months. All three students are used to doing very well in math class and in school generally, but today they do not start their inquiry immediately after the whole-class introductory part of the lesson ends.

Bz: [Benazhir] I'm so unlucky. I'm getting my report card tomorrow. And I'm getting the lowest marks in my life tomorrow. . I'm getting like the lowest marks in my life tomorrow. I'm so unlucky. I told her [her younger sister] that she was happy. She just loves to like make points with my parents.

Lr: [Lorna] xx.

Bz: Yeah. Then they can get mad at me.

Rs: [Rose] If it was your brother's birthday, xx.

Bz: Once in a while, [laughs]. I always forget though, that's my problem. I'm a very busy person out of school.

Lr: I know.

Bz: You don't. I hope you never find out either, but, I'm a very busy person. I don't remember dates. I remember dates but they have to be important. Like my brother's birthday, right,

Rs: You know what this is, this is very weird. I don't know why like, when I go with homework, you know, I have so much to do that I don't know what to do.

Lr: I don't know how either. Now what should I pick?

Rs: Yeah!

Bz: So then I always go by classes. Whatever class I get first, the next day. . .

It is delightful to hear these students talk about the homework which they do so faithfully. The conversation is suggestive of many issues associated with schools and learning. For instance, we can hear hints that Rose's and Benazhir's underlying perspective of homework is of something to get done and we hear little to suggest that homework is for learning or understanding. One might also make comments about Benazhir's expressed sense of efficacy, ascribing her declining grades (grades which for the most part an average grade nine student would be eager to receive) as "unlucky." Also, this conversation was typical of all of Benazhir's conversations with school friends in its topical boundaries. Benazhir eagerly talked about herself and her school life with these students whom she liked and thought of as classroom friends. However, she did not talk about her parents or her relationship with them or the activities she did after school with them.

Do Benazhir's language choices provide an indication of her conception of doing well or poorly? Here, even in terms of sibling relationships, Benazhir's sense of success was tied directly to remembering. It is difficult when looking at transcribed conversation to decide the relevance of different elements to students' learning of mathematics. Yet all of these interpretive claims can be considered as hypotheses about these three students, starting points for considering what personal factors affect, influence, and interact with students learning algebra.

In fact, there is no hint in the conversation to suggest that today's topic is introductory algebra or mathematics of any kind. When looking at a single piece of conversation the field of vision is too constricted to provide a sense of context. What is the situation that generated this small sound-bite?

The date is February 1, and since the beginning of September 22 grade nine students have participated in a small-group interactive inquiry approach to learning mathematics during about one third of their mathematics classes. At the moment, nothing that is happening appears to be different from traditional instruction. Today, the class is directed to start a series of learning sheets after a brief introduction. Although the students know they are expected to do their learning in their groups of three or four, they each have their own sheets to be completed by providing answers to mathematics questions. The students of this class are accustomed to good marks in mathematics -- in fact, marks are very important to most of the students in the room. The tape recorder which a member of each group started at the beginning of the period is virtually ignored by all the students.

Like the other students in the room, Lorna, Rose, and Benazhir have had little but quite traditional styles of mathematics instruction, although by any traditional accounting their mathematics teachers would range from competent to excellent. The first observation which might set these three girls apart from their neighbors is that they interact considerably less than the other groups. During the whole-class portion of the lesson, they did not volunteer with answers or questions and they gave very brief answers if called on. They are quiet learners, but they are not all alike.

Today's lesson is an introductory one about polynomial arithmetic and factoring. The previous day, three classmates, who had chosen during the special periods to do more textbook work than other kinds of activities, had tried to give a formal lesson to introduce algebra to their classmates. In the lesson they presented definitions with examples for the terminology of algebra, including *variable*, *coefficient*, *term*, *constant term*, *degree*, *monomial*, *binomial*, and *trinomial*, and their classmates dutifully copied them into their notes. When the teacher-students arrived at *like terms*, they prepared to give practice questions on adding polynomials by demonstrating with a plastic polynomial kit. The kit had square and rectangular tiles of particular sizes and colors, to be designated as x, y, x^2, y^2, xy, and 1. If turned over, each tile was a second distinct color, to represent that tile's negative amount. The teacher-students had their classmates write into their notes what each color represented, along with a definition of "like term" and the rule that only like terms could be added and subtracted, and then assigned some adding questions from the textbook.

The lesson which the teacher-students provided was a classic example of a traditional lesson from within the delivery metaphor. It also provided a clear example of the great diversity of expectations in a mathematics classroom for what instruction should look like and what learning should be. The three students who led the lesson were flabbergasted by all the questions their peers asked. Because they had not anticipated that those terms and processes which comprised their lesson's content could (and must, in the opinion of most of their classmates) make sense and be purposeful, they had not anticipated all the questions trying to connect the distinct elements in the lesson. The teacher-students had expected that the students would simply accept what they said and write down what they were told because it was sufficient information to do the assigned questions. Only a few students (including Benazhir, Rose, and Lorna) met this expectation of quiet acceptance. The three teacher-students agreed to reorganize their lesson and to provide a more structured view of the textbook's view of algebra at a later date. As researcher-teacher, it was now my task to encourage sense-making about how and why algebra works as it does, simultaneously with the students learning to perform the operations practically. If successful, the teacher-students' second lesson on the particularities of the terminology would make much more sense. Five math periods later, this did indeed come to pass.

Today, however, there has been only a brief introduction, starting with each student cutting out some simple algebra tiles from one of two photocopied sheets to create a personal "algebra tiles kit." (See Figure 1-1, part 1. Appendix A provides a kit for the reader's use.) Using the kit's "unit squares", the students built rectangles with *8*, *9*, *10*, and *11* units, and revisited the simple terminology of number theory (prime, composite, factor, product, square). While this quick teacher-centred lesson seemed to be emphasizing old terminology and completely ignoring polynomial terminology, it was also establishing the context that a rectangle's area can be counted, and can also be confirmed by counting length and width and multiplying. The students sketched these simple rectangles and recorded their dimensions, recognizing that it is useful to say a rectangle's area in multiplicative form as an alternative to the numerical value. For example, to explain why *9* is a square number, students found it sensible and easy to point at a sketch and say, "See, it's three by three. Three squared." (See Figure 1-1, part 2.)

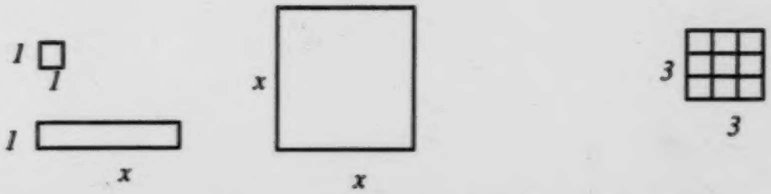


Figure 1-1. The kit's pieces: a unit square, an *x* strip, and an x^2 square; nine as a square.

When the students were asked to build a row of unit squares the same size as one of the kit's "strips", they encountered two complications. Although the strips were the exact width of one unit square, the length of a strip seemed not to be an integral number of units long. Quickly the students also noticed that their neighbor's kit of a different color had strips which were a different length than their own. They agreed that it would be sensible to call it a variable name, and accepted the teacher's suggestion of "*x*".

It would be presumption to hope that the students could immediately see the validity for naming a strip "*x*" because its dimensions measure one in width by an unknown and non-constant length. The concept of variable is not easily developed. (This claim will be re-visited in Chapter 3.) In fact, the kit in its own way takes the variable and replaces it with a given constant by awarding it a meaning which is a given, if unmeasured and uncounted, length. It was my hope at the time that the presence in the classroom of kits where the unmeasured length was a different amount would help students to eventually carry from their exposure to the kit a more complex view of "variable" as representative of many values as well as of an unknown value. However, the algebra tiles are better used for exploring the arithmetic of polynomials than the possible meanings of variables. It makes more sense when using a manipulative to focus on the experiences which the manipulative can support than on the manipulative's limitations.

That is precisely what followed. The students were led to make rectangles of $2x$, $3x$, and $4x$. Just as with the numerical rectangles, they created and sketched rectangles and counted, recorded, and said the dimensions. For example, $4x$ could be recorded as "$4 \times 1x$," "$2 \times 2x$," or "$1 \times 4x$." (See Figure 1-2.) Students stated the dimensions of the

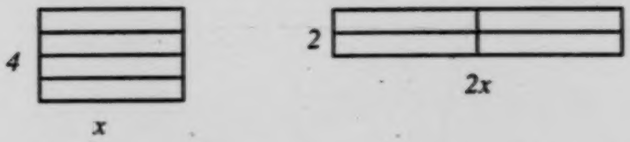


Figure 1-2. Two rectangles for $4x$, showing lengths and widths.

second rectangle "two by two *x*," "two times two *x*," "two *x* by two," or "two *x* times two" without attending to any possible distinctions in meaning. Finally the large square in the kit, with both dimensions matching the length of the strip, or *x* by *x*, was named as x^2.

The lesson so far has taken about fifteen minutes, focusing on algebraic expressions using numbers and the variable *x* and engaging the students in building and describing rectangles rather than memorizing terminology and definitions. Up to this point the teacher has led the activity but it is time for the students to continue on their own. Each student received a set of five learning sheets intended to guide them to make sense of polynomial multiplying and factoring of binomials and trinomials in a single variable. (See Appendix B.)

The sheets were designed to guide students in developing a base of experiences with polynomials as written objects and rectangles as physical objects without relying on a teacher to make clear each step of each process. The sheets also encourage students to make mathematical sense of their experiences with the rectangles they build and the expressions they write. Students respond to each series of similar individual arithmetic questions by building rectangles of the requested size and counting their dimensions. They sketch the rectangles and record the product of the dimensions as being equal to the area specified by each question. There are also open questions to explore or explain various aspects of the patterns that come forward in each series of examples. As the students make connections between the two kinds of objects and the actions they perform with each, it is hoped that the connections will clarify how polynomials work.

If the research purpose was limited to asking how well the algebra tiles and sheets work (or, more generally, how well interactive small-group inquiry works), then the methodology could be as simple as post-testing and the comparing of scores with a control group. If the research purpose was to document and interpret how the students learn well with the algebra tiles and sheets, the methodology would be to observe and describe qualities common to all or most students. However, this research assumed that the uniqueness and diversity of the students was likely to be relevant to how each student learned. It also adopted a proactive stance in relation to factors which affected how each student learned, assuming that interacting educationally with the students while they were learning could affect those factors positively. Listening to the students as they learned became a starting point for both the instruction and the research. This chapter's format reflects this research stance, adopting an immersion approach to locating readers within the context and the process of listening to learners.

This study attempts to validate the claims that classroom mathematics instruction can and should bring to light the factors which affect how students learn well, and that the factors which affect how students learn well can and should be viewed as dynamic processes, open to change through educational interactions. Much more will be said about these issues, and the next chapter will restate the study's purposes in different terms.

Listening to learners

The transcript portion below follows immediately from where the opening transcript portion left off, and with the intervention of the classroom teacher the conversation of Lorna, Rose, and Benazhir turns immediately to beginning the task at hand. [A little coding guide: "MrM:" is me, Mr. Mason, the teacher researcher; "MsL:" is Mrs. Larkin, the regular classroom teacher. "xx" represents an inaudible word or two, while "xxx" is an inaudible phrase or sentence. Extra periods each represent a pause of about a sentence-length.] The students have not been told precisely what to do to begin the first sheet of the algebra tiles activity (Appendix A and B), and so they begin by wondering about how to engage. For Benazhir in particular, her statements suggest the significance to her of being shown precisely what to do, of remembering and forgeting what has been shown to her. All three students make apparent the natural desire for a specific well-defined meaning for the variable *x*, whether it is a color from a set of plastic tiles or a single length of paper strips.

MsL: [Mrs. Larkin] Are you guys not going to do anything today? . Sheet one? Get your, you can get algebra tiles too.

Lr: [Lorna] Okay.

MsL: Or you can use your own, the ones you made. Whatever you want. [Mrs. Larkin leaves.]

Lr: Okay. . Let's use the tiles.

Bz: [Benazhir] I don't understand.

Lr: You go get the tiles.

Rs: [Rose] What tiles?

Lr: The other ones. [the plastic tiles from the previous day's lesson]

Bz: Cause I don't want to use these, paper. . It could equal anything.

Lr: It depends on what *x* is.

Rs: Yeah but we already decided what *x* is, remember?

Bz: No, I don't remember. xx

Rs: No, the green is the *x* and the yellow is the *y*.

Bz: This has to be with polynomials?

Rs: I don't know.

Bz: I forget what, no, yeah, I forget what *x* was.

Rs: *X* is green.

Lr: This is *x*. This is *y*. This is, *x squared*.

Bz: So what is *x*? What is *x*?

Lr: Just *x*, there's no number.

Bz: How do I like, answer, that? . . . . I forget what we did. I forget everything.

Lorna's broad statements about what the nature of variables might be (what *x* would equal with the paper kit "depends on what *x* is") did not help Benazhir. Benazhir felt that *x* was given a value (and a color) the day before, and wasn't ready to consider that it might be something else. Rose was willing to consider that yesterday's colors would be worth discussing but she declined to clarify for Benazhir whether yesterday's main word, *polynomials*, legally covered today's sheet. Today's lesson certainly did not make explicit any possible connections between yesterday's student-led terminology lesson and today's tasks.

In the face of this ambiguity, Rose and Lorna started to explore the first questions on sheet one. They made rectangles, determined lengths and widths, and were sketching and labeling while Benazhir was trying to recall how to code the *x* in the first question. When she asked Lorna once more what *x* is, Lorna's conceptual response about what a variable means didn't match what Benazhir wanted at all. She wanted to be shown precisely how to do the first question.

Distinguishing ways of learning

There is more at issue in this last transcript than what a variable is, even though as a math educator it is very tempting to make that the main theme in this brief interpretation. More significantly, we have access to a deeper issue: Rose and Lorna dived in to the questions (Appendix B), quickly found something that seemed to make sense and proceeded to make more sense of how the questions worked in general. Benazhir thought that being a math student was simply about getting a right answer when asked, and to that end she believed that her primary tool was her memory. Benazhir wanted to remember the procedure for the question before beginning it, believing that her task was to replicate a teacher's demonstration of the procedure with similar questions. There was no apparent point to task ambiguity in Benazhir's perspective of how to be a good student. If something is unclear, it is either attributable to the teacher's failure to make the procedure clear or to the student's memory. Benazhir was not sure whether the teacher's part of her expectation had been met but with the apparent success of her colleagues in getting started she assumed that likely the fault was hers and her memory's. It was not: the whole-class introduction had shown how to build rectangles with unit squares or *x* strips separately, but they had not been shown directly how to handle expressions which required the use of both shapes at once.

In the minutes which followed the previous dialogue, Benazhir quickly got the idea of the activity by watching Lorna and Rose do the first few questions. The sketches which the girls put with their first few questions made the task self-evident (See Figure 1-3), and Benazhir proceeded to catch up. She chatted about French class while checking her

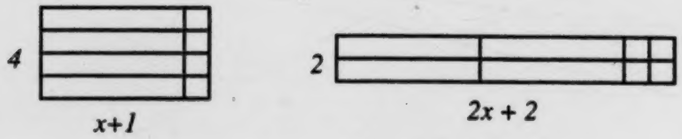


Figure 1-3. Two rectangles for $4x + 4$, showing lengths and widths.

answers repeatedly with her partners. Then the questions changed slightly. We rejoin the transcript as Benazhir looks at Lorna's answer to the first question which uses the x by x square. (See Figure 1-4.)

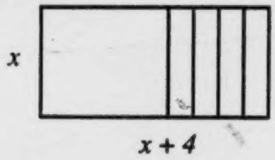


Figure 1-4. A rectangle of total area $x^2 + 4x$, showing length and width.

Bz: Wait a minute that doesn't make a lot of sense. $X + 4$? This is x.
Lr: This is minus x. This is x. And this equals x.
Bz: You put $x + 4$.
Lr: Here's x and then there's 4 on the end, so it would be *plus four*, and then times x because there's one x on that side. .
Rs: Yeah that's right.
Lr: *Plus four* because there's four going down xx.
Rs: Cause like the long one right here does like this.
Lr: Yes.
Rs: So you only plus four because there's only this little one right there.
Bz: *Four x plus two.*
Rs: So it would be like this.
Lr: And then another *two* right there.

It isn't easy for students to recognize and keep track of the way that a square called x^2 has an edge of x, nor that the end of the x-strip is worth 1, matching the unit square, unless they generally understand the structure of the kit as being about lengths and widths and areas all at once. Benazhir could not begin to form that understanding, because she did not perceive the process as related in any way to her prior understandings of multiplication or of area. She did not see that the bottom edge of an x^2 combined with the ends of four x strips worth one each would be $x + 4$, because she was aware neither that the reason for only looking at the edges and ends was to count a value for the length of a rectangle, nor that the purpose of the occasion was for her to notice and make sense of that connection. She viewed the task as something akin to a code-breaking scheme to be memorized--each instruction and each polynomial prompt must be re-coded as a different polynomial according to seemingly arbitrary rules. Did Benazhir view mathematics generally as sets of distinct coding structures, rather than a language structure that makes sense and that makes sense of what the language is describing? Such a stance could make a student's answer generation process not only arbitrary and irrelevant but also quite difficult, since each variation in an instruction, a polynomial, or a rectangle requires a new code-breaking structure. For us to make such a broad conjecture about Benazhir's stance as a mathematics student would be premature: Benazhir learned to generate the expected answers for this type of question and carried on.

A few minutes later, Mrs. Larkin noticed Rose, Lorna, and Benazhir wondering what the instructions meant for the questions without sketches. The transcript below reveals that Lorna and Rose had a multiplicative sense of the procedure already, and easily connected to the new command vocabulary. Rose understood that it doesn't matter which way the two factors are ordered, because it's multiplication and in multiplication the value is the same regardless of the factors' order. In contrast, Benazhir wanted to learn as an arbitrary rule which one is preferred first and which second. Benazhir's partners were assured that they were probably doing things right because the things that they did to make answers made sense. For Benazhir this assurance wasn't available or expected from within the question-answering process. It was only someone who controlled the answers who could assure her that her answers were right.

Rs: 'Write in factored form'?
Lr: I don't get that.
Bz: What's factored form?
Rs: Yeah.
MsL: [pointing at the answers Lorna and Rose already have written down] That. Okay? Normally we write the number in front.
Rs: Oh yeah. [See Figure 1-5, part 1.]

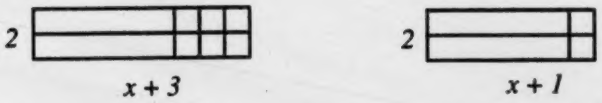


Figure 1-5. Rectangles of total area $2x + 6$, and total area $2x + 2$, showing length and width.

MsL: See. *X plus three times two*. Usually you would see it as *two times x plus three*. This is one side, this is the other side. That's factored form. You just learned factoring. Most people spend three months in grade ten on it. You know how to do it now.

Lr: Sort of.

MsL: Sort of! [laughs] Okay? So, if you had to look at *two x plus two*, there it is, here's two x, all right, what's this dimension? [See Figure 1-5, part 2.]

Rs: *X, plus one*.

MsL: Right, one side is *x plus one*. The other side is *two*. The way you write that is *two, times x plus one*.

Rs: Oh okay.

Bz: So, just the way we have it put the number in front.

MsL: That's right.

Rs: Oh.

MsL: Okay. And you don't have to write a times sign because your brackets already mean multiply. All right?

Rs: Mm-hmm. [Mrs. Larkin leaves.] .

The next portion of conversation follows immediately from the preceding one. Benazhir convinces Rose to monitor her answering of her next question, $2x + 2$. (See Figure 1-5, part 2.) She successfully counts the length and width which are $x + 1$ and 2, and writes it correctly for Rose's approval, but (significantly) she fails to use multiplication vocabulary as she speaks her steps aloud. In the next case, she uses multiplication vocabulary properly, but even as she reads the question she fails to speak the addition component, and continues to say her written answers in ways that suggest she is unaware that the method of recording them is directly connected to multiplication and addition concepts.

Bz: Wait wait wait wait. We have *two x plus two*, right, that's the way we diagram.

Lr: xx

Bz: So that's *one x plus one*.

Rs: Yeah.

Bz: *Two, to the one x plus one*. Right?

Rs: Yeah. .

Bz: [See Figure 1-6, part 1.] Do you have to like add these things? . So it's *three times, one x plus two*. . How would you do *four x two*? [See Figure 1-6, part 2.]

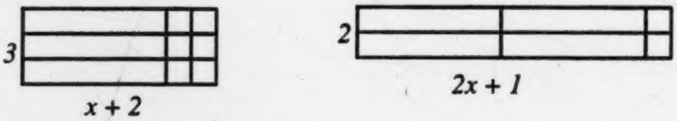


Figure 1-6. Rectangles of total area $3x + 6$, and $4x + 2$, showing length and width.

Rs: It would be like this.

Bz: Oh right. So it's *two x plus three, times,*

Rs: Okay. . .

Bz: [See Figure 1-7, part 1.] So it would be *six x plus four*, so it would have to be this way like this.

Rs: Yeah.

Bz: So it will be *two*, and it will be

Lr: *Three x*

Bz: *Three x plus two*. All right we're on a roll! I feel like a frog.

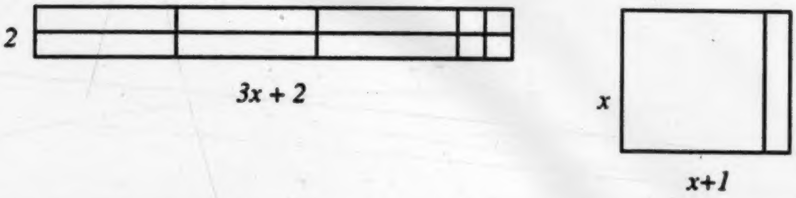


Figure 1-7. Rectangles of total area $6x + 4$, and $x^2 + x$, showing length and width.

I don't know what feeling like a frog feels like, and unfortunately, I have never asked. It is sufficient, I believe, to picture a frog with a smile. Benazhir felt good that she was capably encoding a correct answer on her page for each question. She may even have perceived some internal consistency to her counting procedures and the codes with which she recorded her answers: in the last example, it is not hard to "see" each row as three x-strips and two units, and to see the two rows. She also avoided falling significantly behind her partners, in part by occupying them in helping her with her work.

The next portion of transcript comes a few minutes after the last. Rose draws the others' attention to the question she is on, which again represents a slightly different kind of rectangle. (See Figure 1-7, part 2.) Lorna recognizes it, for she has been working ahead of the other two, and monitors Rose as she gets the length and width. Rose knows that her answer seems right, and can confidently explain to Benazhir, who didn't consider trying it herself first. Benazhir can imitate Rose's words and steps, but without really understanding: note that in this rectangle there are not x distinct countable rows each of $x + 1$ to match the factored form as there were in the last type of question; there are only edge-measures of x and $x + 1$. However, it is pleasing to see Benazhir trying to incorporate the new word, *factor*, into her working vocabulary. Connecting the appropriate student behavior to the teacher's instructional vocabulary is of great importance in Benazhir's scheme of being a successful student, for that will enable her to know *which* memorized procedures to perform *when* on any upcoming test.

Rs: I'm doing this one.
Lr: *X squared plus x.*
Rs: So it will be, *x, x plus one, times x*. All right.
Bz: Huh?
Rs: This is *x*, and this is *one*, so it's *x plus one, times x*.
Bz: It's like factoring right?
Rs: But now you're using *x*.
Bz: So, wait, okay, so this is *x*, so it's *x*,
Rs: And this is *one* right there.
Bz: *Plus.*
Rs: *One.*
Bz: *Plus one, times x*?
Rs: Mm-hmm.
Bz: Okay, what did I just say?
Rs: *X plus one, times x*
Bz: *X plus one, times x*. If you factor it it will be like this.
Rs: Yeah, that's right, good job.

Bz: Okay.

The group continues as above. Sheets one and two provide a few algebra expressions of each kind as prompts to build and analyze rectangles before there is a subtle shift in the appearance of the expression. At each of these shifts, Benazhir is stuck, for her process from the previous questions doesn't work. She doesn't know why it doesn't work, and she doesn't know what might work, and she doesn't even know that she should explore a little to search for something that would work. However, she does know when a process stops working and every time it does she engages with her groupmates. Her groupmates benefit from this to a degree, by getting to go slowly through the process verbally for a given example. However, they do not get to engage in any sense-making descriptions of a general nature or statements of overall procedures or justifications, for that is not what Benazhir requests or desires. Nor do any of the three of them initiate any discussions about how they are learning or why they are learning in this way. This deficit limits all three students but whether such talk by Rose and Lorna might have triggered Benazhir to productively adjust the way she is approaching the learning opportunity would only be conjecture.

Deciding what to learn

We rejoin the transcript when there comes a slightly different transition point. When the questions shift slightly to include two of the x^2 squares along with some x strips, Benazhir realizes once again that she cannot use the method she used with questions with only one x^2 square. However, from trying to get the answer by her previous method she does generate a possible kind of answer, even though it's not what she finds that Rose has. (See Figure 1-8.) Rose's answer is clearly the one the teacher expects but Benazhir shows hers to Rose before dismissing it from her mind.

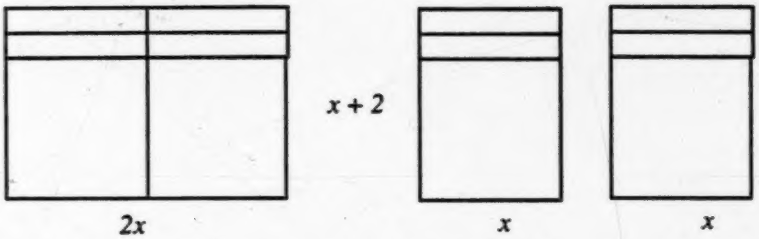


Figure 1-8. Rose's standard version and Benazhir's two-section version, both $2x^2 + 4x$.

Bz: What is that, x, *times, two*?

Rs: *X plus two times, two x.*

Bz: Yeah, but that's what I mean, like this will be *x by x plus two*. .

[long silence. Rose and Lorna are proceeding smoothly. Benazhir is falling behind.]

Bz: Why do I have, is that right?

Rs: It's supposed to be one rectangle.

Bz: Yeah but, then, look , this is what I did, I just figured this out , and then I put a two in front.

Rs: Yeah, but then, but then, this has to be, this is x, and if you put a two in front then it would be *two x*. It should be on the outside of the bracket.

Rose could tell that likely her answer was the expected one, but she could also see that Benazhir's answer should be okay. Rather than one big rectangle, it had the correct total area shaped in two rectangles of dimensions $x + 2$ by x. Doubling is a multiplication idea too, so Rose suggested that Benazhir just needed to clean up her version by writing the 2 which represents the doubling so that it would not look like $2x$. Rose suggested changing $2x(x + 2)$ into $2(x\,x + 2)$ but she did not see how this change would complicate the reading of the length times width section of the answer.

Although Benazhir was not interested once she recognizes that it was not the teacher's expected method, Rose recognized that factoring is about creating multiplication statements, and length times width is one way to do that, but making two identical things or groups is another way. One student is concerned only with what the teacher expects while another is concerned with what makes sense to her. On the next question, $2x^2 + 6x$, Rose finds the standard answers, explores Benazhir's idea, and calls to me. (To follow Rose's thinking, add two more strips to each answer in Figure 1-8.)

Like any teacher might, I get them to take me through the standard two solutions, so I can hear them explain the dimensions. Of course, they do well. Rose then asks me to look at her two-rectangle version of this question. Again, I have them say its dimensions to me, which goes well, although at one point, Benazhir says, to describe $2(x+3)$, "*X plus three*, like, *two*." When I can see what the new conception represents, I help them record a factor of 2 separately from the rest of the factors by using square brackets, $2[x(x + 3)]$. "So inside the square ones you have x times $x + 3$, and then once you've done the inside brackets you double it. Pretty fancy! That's pretty good. Actually it was the other two answers I was expecting you to get, but that's actually a fancier, prettier answer."

I am provoked mathematically by Rose's conception (only much later do the tapes show the conception's true source to me) to consider all the other questions we have factored. Could they be made into separate piles instead of length and width? Although all the monomials and binomials using unit squares and/or *x* strips can be viewed in this other standard conception of multiplication (for little is needed to convert a stack of equal rows into a series of equal piles), few of the binomials including x^2 (for example, $x^2 + 2x$) can be done that way, unless you think non-concretely (with that example, one could imagine creating *x* piles with $x+2$ in each pile). It is a re-viewing of the mathematical relevance of factoring with algebra tiles for which I am grateful: I have been led to integrate a sense of counting distinct groups with my prior sense of measuring rectangles' edges. As with numbers, polynomial multiplying can be considered as a multiplicative array or as repeated addition of equal amounts, and the kit allows for this.

Benazhir's opinion on the possibility of more than one right answer went unexpressed. We can tell that she did not see that the answer-making she was doing could be related to the counting or adding or multiplying with numbers which she had been doing for many years. For almost all the rest of the class that day, she alternated in her oral description between multiplicative ("So that is, *x*, that is, *x by x plus two*" for $x(x+2)$) and exponential phrasing ("*One x times two, to the third*" for $3(1x + 2)$). Benazhir needed some mathematical language to point toward the questions and answers she wanted help with, but did not seem to consider as relevant developing a precision with her oral use of mathematical language. Written expression is the traditional format for students' answers, and so Benazhir's meticulous concern for detail in her written work did not transfer to her oral statements. Nor did Benazhir show any desire to understand the connection of factoring to her prior experiences with multiplication.

As we rejoin the transcript with ten minutes remaining in the period, the three students are working together to help Benazhir finish the last questions on sheet one. To Lorna's supportive suggestion, she responds in a way suggestive of her interest lying solely with getting correct answers onto the sheet.

Lr: Want to do it the fancy way?

Bz: I find it easier. I'm sorry I didn't think it was the fancy way, I found it easier just to separate them and put a two ~~in~~ front. That will be *x by x plus one.*

Rs: Now, put it the way he put it, that would be like, like this, with square and round brackets.

Making connections

With five minutes before the period's end, the three students are a little concerned to notice that most of the other groups have begun sheet two, or are even half way through it. The other groups also have been engaging in conversation while doing the sheets (about the mathematics and otherwise) to varying degrees and for various purposes but generally they are slightly farther along. Every group is given the option of assigning themselves some of the sheet as homework but they also have their regular homework which is an optional reflective question. Despite understanding there is no pressure from the teachers to push forward, Lorna, Rose, and Benazhir wonder what to do about the sheets while trying the first example from sheet two, their first exposure to a trinomial, $x^2 + 2x + 1$. Since they can make a rectangle like the ones on sheet one from the first two terms, positioning the unit square is difficult to visualize. (See Figure 1-9, part 1.)

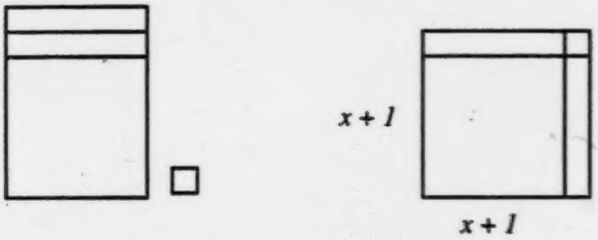


Figure 1-9. Getting stuck on $x^2 + 2x + 1$, and Lorna's successful idea.

Lr: Why can't we get this?

Rs: We're stupid.

Bz: No, I'm stupid.

Rs: Well, we didn't find another way to do it.

[long silence. Lorna and Rose experiment, changing the pieces they use, while Benazhir watches.]

Bz: Let's not do this for homework!

Rs: Well, everybody's on the second sheet so we better do it too.

Bz: I can only do like this.

Rs: Let's see, x squared, plus *two x* plus what?

Bz: How would you put *one*?

Rs: Mr. Mason!

MrM: Hi guys.

Rs: For number A we already did this.

MrM: Okay, what I see here when I look at your piles, is that what you want me to look at? Is that what I should look at?

Rs: Yeah. xx. .

MrM: *X squared plus two x plus one.*

Rs: Yeah, but where will the one go?

MrM: Yeah that's the problem isn't it. If we just add it on to the end, that's not a rectangle so we can't do length times width. Hmm.

Lr: Could we, xx. [She moves a strip to the other edge, and fills the gap where the two strips do not meet with the unit square. See Figure 1-9, part 2.]

MrM: Yes, we have a rectangle!

Bz: Oh, what a job!

Rs: We figured it out!

Lr: Is that a rectangle or a square?

Rs: So it's x by x, so that's like *two x, plus one,*

Lr: Oh, well, a square is a rectangle, so.

Bz: So now wait, *x plus one, squared.*

Rs: This is weird. Mine is closer to that. How does it go?

Bz: Mine is *x plus one squared.*

This last line is not just a success story for the three students regarding their first trinomial. Although Lorna formed the needed rectangle, Benazhir has surged ahead with it, leaping beyond Rose's temporary difficulty counting the length along the edges. Also, probably aware of the significance of Lorna's observation that this rectangle is a square, Benazhir uses exponent notation ("squared") to express her answer, just as any of the students could for the example from the period's opening (see Figure 1-1, part 2), of *three squared* being a rectangle for nine. From this point on, Benazhir's use of additive, multiplicative, and exponential vocabulary whenever she stated factoring results orally was completely consistent with the other students. Because this success in her group to provide and express an answer for a new kind of question falls at the time the consistency started, I suspect that this was the moment when she saw that in the algebra kit environment, the big idea of getting factors was thinking of length times width for a rectangle, just like it had been for "Nine can be three squared." We have a glimpse of a connection being made.

Benazhir's learning

What understanding of how students learn mathematics can we take with us from having met Benazhir? In the terms of Belenky, Clinchy, Goldberger, & Tarule (1986), Benazhir seems to have been a clear example of a "received knower". Received knowers "equate receiving, retaining, and returning the words of authorities with learning -- at least with the kind of learning they associate with school" (Belenky, Clinchy, Goldberger, & Tarule, 1986, p. 39). Benazhir's concentration on the teacher's key words (*polynomial, factored form,* and *squared*) is recognizable as a part of this stance. Although Benazhir talked comfortably for social reasons, she did not use her own words to describe mathematics ideas even while working out how a particular question was to be done. Benazhir asked her colleagues for explanations but did not re-state received explanations in her own words. This contrasts with Lorna and Rose who did restate concepts, something we might naturally expect of student who are trying to make sense of a concept. For examples, consider Lorna's extended explanation of what *x* means and Rose's description of the double-rectangle version of some questions. Although none of the three students explicitly addressed issues of how to learn in the conversation on the day discussed here, their words and actions showed that Rose and Lorna had expanded how they learned beyond received knowing. They actively made sense of the questions they posed for themselves as they proceeded through these learning sheets. Benazhir did not.

However, this description ignores totally a significant element in Benazhir's math learning story. To categorize her approach to learning or to focus on what she did not do belittles the processes by which she achieved her success in school mathematics. Not only was Benazhir meticulous to assure that the technical words of her teacher were matched to her perception of procedures, but she successfully carried a large number of separate procedures in her active memory each matched to particular question types. As happened in this chapter, procedures could come to be connected to previously understood concepts, but this was not Benazhir's cognitive intention as she managed the separate kinds of questions. Instead, she was committed to the achievement of a non-conceptual style of automaticity by repeated practice and remembering the features of the instructions and the questions that would call for a particular procedure. This ability of students such as

Benazhir is simply impressive: to continue to work and generate answers to example after example without having a conceptual framework to obtain answers or to imbue the work itself with its ostensible value for developing understanding.

We might have considered Benazhir's belief in remembering as being a simplistic stance regarding how to learn. Yet it is actually what so far into grade nine brought her academic success. She was able to remember the specific procedures for each variation of factoring question she encountered, learning each procedure from others. She did this successfully despite not perceiving how the procedures could make sense to her. In fact, her sense of learning meant that she did not even expect these procedures to make sense. It is even more remarkable that Benazhir did this despite not being able to see the connections between the algebraic arithmetic answers she was writing and speaking and the numerical arithmetic she understood. This last omission ended however when she encountered a circumstance where specific exponent vocabulary she already knew made sense of a new question. Did she re-view her previous mistaken use of exponent language as mistaken and recognize the role of multiplication and addition vocabulary in its place? This is hard to say, since nothing of that kind was verbalized: it is one way to suppose that Benazhir reoriented her language to use terms which were consistent with the arithmetic operation being described. In the absence of talk about how Benazhir saw her own ways of learning or ways of perceiving mathematics itself, we must be careful with our suppositions.

Should we view as significant the absence of such talk? In a research sense, the absence is significant in that it leaves us to surmise from her mathematical actions how Benazhir sees her role and purpose as a student. Similarly the absence of such talk leaves us able only to hypothesize, if we accept a developmental or stage-based perspective, about how Benazhir has maintained an approach which is much less suitable for complex topics and inquiry-style learning than the sense-making approaches of her two groupmates. Perhaps the absence of such talk is a significant part of the answer to this question: a student's options for re-viewing and amending the ways she is approaching learning and mathematics (or even of thinking about these concepts at all) are significantly narrowed if there is no discourse about them.

The ultimate success by Benazhir in coming to make sense of the multiplication-area structure of both the algebra tile rectangles and the notation is not something we could expect of many adolescent mathematics learners. Nor is Benazhir's status as an excellent school-math student the norm for received knowers. To help Benazhir carry her success in mathematics forward, the question which must be addressed first is whether to consider her orientation toward received knowledge as a given to which curriculum events must be shaped, or as a starting point which curriculum can assist her to grow beyond. I think that efforts to shape curriculum to received ways of coming to know must founder, as the elements to be learned become more complex and more abstract, leaving behind many potentially capable learners. This leaves us to consider how curriculum could address the encouraging and supporting of students as they grow beyond received-knowing ways of learning, and that is what much of this document addresses.

Toward future chapters

I hope that this interpretation of one session of three learners shows both the challenge and the possibility of research processes that accept an intimate stance within the classroom context. Benazhir's learning showed its personal nature in two senses of the word. First, learning varies significantly person to person. The processes by which Benazhir came to understand the factoring of polynomials were significantly different from other students who shared the same classroom activity and who seemed to have ostensibly similar orientations to the activity. Second, learning processes implicate elements which are significant to Benazhir's sense of self, including her understanding of herself as a student and her sense of how to learn mathematics. When a curriculum change encourages students to learn differently from how they are already learning, students' adjustments to the opportunity will be specific to each individual and will involve significant aspects of each student's identity.

Considering students' approaches to learning as both diverse and dynamic makes research into the effects of curriculum change challenging, but it also defines a window of opportunity for determining the nature of students' interactions with curriculum changes. Each personal element that complicates a student's learning processes provides an observer a window for noticing the nature of the student's learning. Also, each of those elements are an opportunity for a teacher to affect that learning through curriculum and instruction. To open the personal aspects of students' learning to influence and observation, teaching and research should be positioned within interpersonal relationships with the learners. The instruction and research described here took such a stance.

What would be the best mathematics curriculum for Benazhir? To answer this question, first we must decide whether to consider as static Benazhir's orientation toward her learning or consider that orientation to be something which curriculum and instruction could affect. The rest of this document is an exploration of the latter stance: **What could mathematics curriculum pursuing student understandings of pre-formal algebra look like, and what would be the nature of the students' learning, if students' orientations toward learning (and the personal factors which influence that orientation) were viewed neither as irrelevant nor as a given constant, but as a learning space to be developed concurrently with the learning of algebra?** Benazhir and her classmates will provide a particular answer.

Chapter 2. A Map for the Reader

The structure of this document may differ significantly from what readers may expect. This chapter's main purpose is to assist readers in establishing a more comfortable relationship with the text. This means that its role lies outside of the unfolding of the account of events which is the purpose of the other chapters. One of its purposes is to document details which were important to the research as a process, but which did not become elements of the story of the research. Its other purpose is to step above the story of the research, and provide a guide for readers to the document itself. Described in turn are the context of the study, the nature of the instruction, and the nature of the research. The chapter closes with a guide to the chapters which are to follow.

This chapter was not placed as a preface because its intention is easier to understand when the reader has had an opportunity to perceive the nature of the study and the writing about the study. If placed as an afterword, this chapter would not prepare readers for the purposeful changes in focus and style which occur from chapter to chapter. The first words of a story should invite readers into the flow of events and invite them into relationships with the characters. Similarly, the last words should provide closure for the narrative and a sense of completeness in terms of the story's relationships, rather than engage readers with the technical aspects of the events behind the story and its writing. An in-between placing for this chapter allows both the first and the last chapters to be situated within and feature prominently the learning processes and relationships which were the foundation of both the instruction and research.

The context of the study

> Teachers' understanding of context, inside and outside the classroom, is the single most critical factor influencing success in teaching (Adams & Hamm, 1994, p. 5).
>
> I believe that we shall be able to interpret meanings and meaning-making in a principled manner only in the degree to which we are able to specify the structure and coherence of the larger contexts in which specific meanings are created and transmitted (Bruner, 1990, pp. 64-65).
>
> We will be unable to talk about the specifics of instruction in a theoretically grounded way unless we place analyses of learning within the context of classroom

> social interactions. ... Adherents of the process-product approach are clearly trapped on one side of the chasm that currently separates research on learning from research on teaching. Constructivists are in danger of becoming trapped on the opposite side of the same divide (Cobb, Yackel, & Wood 1991, p. 169).

> We challenge the critical social scientist to move more fluidly between theory and classroom life to understand more richly how the power relations in classrooms unfold in routine activity (Gutierrez, Rymes, & Larson 1995, p. 448).

The study described in this document took place in a grade nine classroom with 22 students in an urban grades one to nine school with about 140 grade nine students. The class was chosen because the teacher and I had a research relationship established over the previous two years. I first met Mrs. Larkin (a pseudonym, like all names in the study) while serving as a research assistant for studies situated in grade seven fractions (Kieren, Davis, & Mason, 1996). The previous spring I had completed a pilot study in Mrs. Larkin's two grade nine classrooms with a ratios unit structured from a constructivist orientation. Groups of two, three, or four students were encouraged to complete inquiry-oriented labs which fostered the use of ratios in a wide range of contexts. The labs themselves worked as curriculum very well. However, emerging themes demanded our attention both as teachers concerned with the success of our students and as researchers interested in factors which affect student success. These themes included student confidence, autonomy, and sense of purpose as they influenced students' responses to changes in instruction and teacher expectations. Mrs. Larkin and I agreed that during the following school year I would orient the research to address these concerns along with the curriculum design questions which the pilot study had explored. Although the number of periods would not be increased, the instruction for the study would be spread out over a much longer period so that there would be more time to guide and notice students as they changed in terms of the personal factors that affected their mathematical learning.

It was late in June when the administration of the school made a remarkable decision of considerable significance to the study. In response to concerns that some students were bored in grade eight math, an agreement was reached with parents that the five heterogeneous grade nine classes would all be taught math at the same time. This would allow one classroom of students to be gathered together from all the classes to learn more mathematics. The goal that was established was that the students would see if they could complete the grade nine and ten curriculum in a single year. Mrs. Larkin was selected to teach this class and address this ambitious goal.

In Alberta, the grade ten math program has three separate streams. The students in this group were only interested in the top-stream math, so the attempt to do the grade ten curriculum during grade nine meant much more than a double-speed approach. In grade nine and in previous grades, the math curriculum is intended for all students and spirals through topics. However, in Math 10, there is little review of prior concepts and considerably more new material since the curriculum is intended only for quite capable students with complete understandings of prior concepts. This meant that the attempt to do Math 9 and Math 10 in one year represented much more than a doubling of content and pace. It would also require that the students upgrade their learning approaches. From procedures which had allowed them to get very good marks in heterogeneous groups with a spiral curriculum designed for all students, the students would have to develop learning approaches suitable for dealing with instruction designed and paced to challenge students of their ability.

My first reaction to this surprising news was to relocate the study, but Mrs. Larkin suggested that I stay. She pointed out that such a situation could provide significant opportunities to study the themes that had emerged in the pilot study, the personal factors which influence students' orientations to learning when they face changes in the orientation of curriculum and instruction. This possibility made considerable sense to me and ultimately this writing is the result of that window.

I had four concerns for situating the research in such a classroom. First, ideologically I did not feel comfortable with vertical acceleration as a response to student boredom. Second, I was very concerned that the goal of completing Math 9 and Math 10 in the time normally allotted to Math 9 was unrealistic. If students ended up with a weaker foundation for the math courses which would follow or if they took Math 10 anyway the following year, they would be at a significant disadvantage as a result of the process. In the first case they might be unable to maintain the level of achievement they desired and deserved, and in the second case they would again be faced with boredom at a crucial time in their high school program. Third, I felt that the kind of curriculum and instruction I had offered in the ratios pilot study would serve students facing this challenge very well, but it required the time that a spiral curriculum provided and wouldn't serve as well if compacted into an accelerated format. Ratio and proportion was not a significant element in the Math 10 curriculum. Finally, I was concerned with the generalizability of findings which were based on students who had selected to try such a program.

With one simple strategy Mrs. Larkin convinced me to situate the study with this group: she agreed with all my concerns. In fact, she stated that she shared the first two concerns and suggested that this was one reason she felt my involvement would be valuable: when she and the students made the apparently unavoidable shift in their formal goals for the program, a richer understanding of their own purposes and learning processes would be important for them to set appropriate educational goals. Also, if some or all of the students were going to repeat some of the concepts in Math 10 the following year it made sense to offer them instruction in a style unlike the traditional approach they would likely get the following year. She agreed that the ratios unit wasn't central to the grade nine and ten unit, and simply said that I should address a different unit of more significance in the grade ten curriculum.

As for generalizability, she agreed once more, suggesting that nothing done in particular classrooms is particularly relevant to other classrooms or other teachers unless the teachers perceive that it could be transformed to their classrooms. In fact, my concerns for generalizability were residues from when the pilot research had been conceptualized as a teaching experiment to validate the innovative ratios curriculum. When a study's focus changes from a curriculum orientation and design to personal elements such as student autonomy and students' views of how to learn, the study's findings will necessarily be about the particular students being studied no matter who they are. Attempts to deny the situated nature of such a study to demonstrate applicability across contexts would require that the teaching and learning that took place be viewed as if they were not deeply personal and interpersonal acts. I agreed to situate the study within Mrs. Larkin's unique classroom, and to develop algebra activities suitable for that context.

This study is not about the administrative decision to pursue an accelerated program. It only indirectly concerns itself with the students' sense that being less bored by math meant doing math the same way at a faster rate. Generally the students had assumed that they could do less review of prior concepts and make do with briefer explanations, get by with slightly less practice with each concept, and double their pace without much change in their workload. By mid-year all the students had recognized the qualitative shift in amount of curriculum in the upper-stream mathematics courses in high school and anticipated taking Math 10 the following year anyway. Many had recognized that mathematics in school was not a narrow linear pathway where success involves going as far as possible as fast as possible, a recognition which was only part of broader changes in how they came to perceive mathematics and how it could be learned.

The activities which I provided included grade ten concepts, but their emphasis was to provide an experience base without emphasizing the formal content and arithmetic practice which they could expect to receive in a traditional grade ten classroom. As a result, despite the initial orientation of the students, the curriculum for the year as a whole is more fairly described as enriched rather than accelerated. Similarly, rather than categorizing these students by their choice to try an accelerated program, it is more accurate and useful to characterize them by their mathematical expectations. Like other well-motivated students used to getting good marks in traditional junior high classrooms, these students expected to get marks in the B and A range in the academic stream of high school mathematics.

The nature of the instruction

> Because researchers widely agree that a constructive, active view of learning must be reflected in the way that mathematics is taught, classroom mathematics experiences should stimulate students to explore and express their own ideas and build on their past understandings. Students should have opportunities to interpret mathematical ideas, rules, and principles and to construct mathematical understandings for themselves. To do this, students need to become explorers in problem-solving investigations and projects that fully engage their thinking and reasoning skills (Adams & Hamm, 1994, p. 184).

> We are in a position to construct curricula that have continuity and depth and that carry their own reward in giving a sense of increasing mastery over powerful ideas and concepts that are worth knowing, not because they are interesting in the trivial sense but because they give the ultimate delight of making the world more predictable and less complex. It is this perspective that makes me optimistic and leads me to believe that our present flurry is the beginning not of another fad, but of an educational renaissance (Bruner, 1960/1995, p. 335).

> A dialogic, inquiry-oriented curriculum is not just an academic's theoretical solution to the malaise of conflicting educational goals, but an achievable classroom reality, that educators at all levels, and in all subjects can construct for themselves (Wells, 1995, p. 234).

In a typical week, one or two periods of about forty minutes each were allocated to the research component of the instruction. During the other math periods Mrs. Larkin offered lessons on other topics, with whole-class lessons followed by assignments usually taken from the textbook. The instructional component of the research began in early September and continued into March, totaling about 40 periods, with closure conversations with groups of students extending into April. This study focuses only on the instruction which I provided for one or two periods each week, which can be described as dialogic small-group inquiry interspersed with teacher-led whole-class discussions. The curriculum elements addressed within the inquiry component included patterns and functions and the arithmetic of algebra. Later in the sequence, I also provided inquiry activities which dealt with the Pythagorean theorem and the arithmetic of radical expressions, but those lessons are not described in this document.

The students were used to learning mathematics by doing paper-and-pencil practice after a teacher presented any key terminology and concepts and made explicit the specific procedures to be used to respond to particular kinds of mathematics questions. The inquiry instruction had a very different character and expected students to learn in significantly different ways. Whole-class teacher-led portions of lessons helped students to engage with each inquiry and to recognize what was expected in each inquiry both mathematically and behaviorally. After any small-group inquiry, whole-class discussions helped students validate the inquiry processes they had been using and to recognize the understandings they were developing. The whole-class teacher-led portions of lessons were also opportunities to model and encourage the kinds of discourse that would enrich the time students engaged in small-group inquiry. The students were used to discussing concepts to develop their awareness of important procedures, connections, and understandings in other subjects, but they weren't used to it in mathematics class. When an opportunity presented itself, lesson time was devoted to discussing the concerns students encountered with small-group inquiry.

The students became aware of the different nature of the instruction and the different expectations for their activity during the first strand of inquiry activities, "Patterns." In the Patterns strand, groups of three or four students explored, modeled, and analyzed relationships within four natural contexts which yielded mathematical patterns and functions. Although it was of significant foundational value in setting up the strands which followed, and although students learned considerable mathematics during this strand, the chapters which follow do not describe the Patterns lessons extensively. I will describe them briefly here as a way to describe the general style of inquiry instruction.

The first Patterns inquiry, Handshakes, was typical of the activities which comprised the first strand. (See Appendix C and Appendix H.) It involved determining how many handshakes there would be if everyone in a group of a certain size shook hands. After a whole-class session which introduced the question and encouraged students to engage, the students worked in groups of three or four to determine particular values for the given question. They looked for ways that the values that they obtained were dependent on the values that they had known. They looked for patterns in the sequence of answers which they obtained. Also, they expressed what they understood both informally and formally, the latter with mathematical notations such as tables of values, graphs, and algebraic expressions.

Three more such patterns inquiries followed Handshakes. In the final inquiry called Painted Cube each group of students had a context which provided a wide variety of different patterns to explore and compare. In terms of the functions curriculum, the students learned to associate mathematical patterns with ordered pairs of values and express relationships and sequences verbally, mathematically, and graphically. In terms of the algebra curriculum, the students' desire to express their imagery and understandings effectively and efficiently provided a purpose for learning about algebraic notation and the arithmetic that governs it.

In all the inquiries in the first strand, the initial exploration of the question was done interactively, first in a teacher-led session and then in small groups, without a set of prescribed questions to be answered. After that exploration, the students were able to use question sheets which I had prepared to help them identify and express the patterns and functions that they were developing. Over the course of the first strand, the groups learned that it was much more effective to talk extensively about the sheets as they filled them out, but most groups maintained their habit of each student filling out her/his own sheets. Even in groups where students discussed an open question at length, the students wrote personal answers or at least used personal wordings of the group's consensus. Slowly the students learned how to be interactive about mathematics and the benefits of doing so. Slowly the students learned that the purpose of doing the inquiry was not just to get the work done but to figure out mathematical concepts. Chapter 1 mentioned these changes, and they will be developed further in later chapters.

There was one unique instructional method used each day of the study, and during the initial strand on patterns and functions the students explored ways that they wished to respond to this element. No mathematics homework (in the traditional sense of the word) was ever assigned in the inquiry-based mathematics classes. Instead, a "homework" question was asked at the end of the period. (See Appendix G for a list of each day's questions.) The question was an attempt to have the students reflect about some aspect of their learning processes during the inquiry that had taken place that period. This homework was not for marks and the students were told they could write as much or as little as they wished. If they wished to write nothing at all, that was fine. I provided a personal written response to each student's writing, to provide a sense of audience for the student and to make the homework writing more dialogic. This allowed me to make the homework a way to offer support, encouragement, and (occasionally) provocation for the students' attempts to reorient their learning processes and stances. Different students reacted in different ways to this invitation, and all the students explored different ways to incorporate this aspect of the program into their own learning. This element appears in later chapters both as a vital aspect of the instruction and as a source of data.

Other than the Patterns strand, all other strands of algebra instruction receive detailed attention within this document. The strand which followed Patterns involved the negotiation of testing and marking processes to match the novel instruction and learning processes of the students. The students were not familiar with student-centred interactive learning processes in mathematics or with any forms of authentic assessment in mathematics. As a result, they had considerable difficulty with the possibility of being tested and marked in ways that would align with the learning processes they had been using. This strand is a source of rich data about students' orientations to learning, to mathematics, and to school, and is the focus of more than one chapter. However, it was also a strand of significant meaning to the students in that they learned about their own orientations and many began or redirected changes to those orientations during this time. Although the students learned more about functions and algebra throughout this strand, it is in the sense that they learned about their own orientations and expanded them that I would describe this strand as being fundamentally instructional rather than an assessment process. Readers will be able to judge this for themselves while reading later chapters.

Overlapping the assessment strand was a short strand organized around a novel set of manipulatives for making sense of algebraic expressions and their arithmetic relationships. This instructional sequence asked each group of students to describe the mathematical relationships which could be generated from a specific set of grid-paper rectangles. Each group could describe the relationships which they developed by using specific values that were derived from their rectangles. However because the groups each had different sets of rectangles, to prepare for whole-class discussions it made sense to use algebraic expressions to describe the relationships in a way that would apply generally to all groups' rectangles. In other words, it made sense to describe the relationships algebraically. In later chapters this strand of inquiry serves as an example of the application of constructivist principles to the design and implementation of curriculum. Emerging from this activity were important examples of the nature of students' personal orientations to mathematics and learning, because different groups of students interacted with the learning opportunity in very different ways.

Chapter 1 shows the beginning of the final strand of inquiry. It consisted of the study of algebra tiles and the completion of the sheets which Benazhir, Rose, and Lorna were doing. As a culminating strand, it served well as an opportunity to notice differences among students' evolving approaches to their learning. The test which closed this strand provided another opportunity for considering the role of assessment in noticing and influencing students' learning and their approaches to learning, and it is featured prominently in the chapters which deal with assessment .

The purpose of the study

As stated at the end of chapter 1, the purpose of the study is to investigate the question, What could mathematics curriculum pursuing student understandings of pre-formal algebra look like, and what would be the nature of the students' learning, if students' orientations toward learning (and the personal factors which influence that orientation) were viewed neither as irrelevant nor as a given constant, but as a learning space to be developed concurrently with the learning of algebra? In particular, the study examines the role of constructivism as a learning theory to guide the designing and teaching of a curriculum to introduce algebra to grade nine students. The study investigates the challenges which students may face when making meaningful adaptations to reform-oriented instruction in their learning orientations and the challenges for teachers of providing instructional support and guidance to the students as they do so. Students' epistemological learning, ontological learning, and learning about learning and studenting are found to be significant concurrent processes to the students' adapting to reform-oriented mathematics instruction. In the light of these three concurrent elements, the study investigates the challenges of making changes to the teacher-learner dyad within reform-oriented inquiry-based instruction.

The nature of the research data

> Evaluation of complex, cognitive-behavioural interventions requires the measurement and observations of actions and narratives of actions that are context-driven and that flow from the meanings given to them by program participants. Measurements and observations that impose sanitized questions not part of the context and process are untrustworthy. Allow observers to judge what they see in participants' actions, and allow participants to tell their stories about how they have come to understand what they have learned and how they apply their understanding (Campbell, 1995, p. 451).

> Certainly in the beginning stages of an intensive examination of students and reform, there would be no substitute for talking to students directly, in settings where students can express their experiences freely, and without the constraints of an adult-imposed model of the most significant issues. This would argue for a heavy emphasis on qualitative approaches (Corbett & Wilson, 1995, p. 16).

> Teachers will not have a major impact on the way students use their minds until teachers come to know how their students' minds are working -- one by one. Teachers cannot help young people make sense of things if they do not have time to answer their students' questions -- and time to really hear the questions (Meier, 1995, p. 372).

Data during the research instruction came primarily from three sources. First, each of the seven groups had a tape recorder which the students set up and looked after during each period which was part of the research instruction. The students always had the right to stop the tape or even rewind and record over something that was said, but they seldom did this, especially after the first few days. I wore an eighth tape recorder as well. The group tapes captured whole-class lessons well, and effectively captured the activity of each of the working groups during small-group inquiry. Second, the students' reflective writing responding to the daily homework question helped to notice and understand the perspectives of students and to perceive the nature of changes that they were undergoing.

Third, any written products which came from the inquiry lessons were photocopied before being returned to the students. Because of the purposes of the research, students' statements, both oral and written, are woven throughout Chapter 4 through Chapter 8, as they were in Chapter 1.

The fourth source of data comes from recorded "closure conversations" with each group of learners. Each group was invited to schedule interviews with me after the research instruction had ended to discuss their understandings of mathematics and how they were learning it. Originally conceived to be post-intervention interviews to address my needs for research data, the process was reorganized to be built around the questions and opinions that the students wanted to discuss. These interviews were to fill a role more traditionally reserved for exit measures of achievement but in this case "achievement" means the changes in perspectives of the students regarding mathematics, its learning, and their own sense of themselves as learners and as students. In part because the flow of topics was determined by the students, the students' achievement in these areas became apparent in their words. The closure conversations included considerable reflective description of the inquiry program and the learning that resulted, as Chapter 8 will show.

There was one limitation on the use of data within the report. One student did not provide permission for her products and words to be used in the study. As a result, conversation from her group of students is not included here. Also, no portion of whole-class conversation in which she took part or was mentioned was used here. Finally, no portion of her two partners' writing is used here. At first this appeared to be a significant limitation, but educational research has one remarkable advantage over some other fields: there is no lack of data or access to situations which generate data. I believe that readers will find the chapters to follow more than sufficiently rich in data of all kinds for each situation that is explored.

There is one significant loss within the excluded data which I feel is worth mentioning as an ethical consideration. The no-data student and her two group-mates chose not to participate in the inquiry activities for an extended period of time before opting to participate again. First the no-data student (following Handshakes) and then her two partners (after the next Patterns activity) accepted my formal offer not to participate in the research process, and the students were provided with direct instruction and more traditional practice activity for those periods. At the end of the Patterns strand, the three students joined the inquiry-based instruction again (it was this group which provided the textbook-based algebra terminology lesson which preceded the lesson on which Chapter 1 was based). Especially because one of the students wrote to me over an extended time after the research had ended, I have some regrets not being able to include their journey toward perceiving mathematics differently. It must be left to other research to illuminate how particularly reluctant and/or shy learners adjust to mathematics reform activities.

There were two tests during the research instruction. However, this study is not about how to increase student achievement in the traditional sense of that word. The tests were more significant as parts of the learning which the students did than as measures of that learning. This is especially true in terms of the students' learning about what should count as learning and the students' learning about their relationship with the authority structures of the mathematics classroom. It is also true in terms of the students' mathematical learning. Both tests were authentic instructional sequences in their own terms, extensions of the skills and content that were their assessment targets. There was no attempt to make the scores on the tests meaningful in comparison to any external norm, so the tests serve this study only by being part of the instructional relationship between me and the students. Because marks were so significant to these students, the testing sequences generated considerable data for the study through all four of the channels described above. However, the tests themselves were not designed as or used as sources of achievement data for this study.

The decision not to use content tests to generate data about how much had been learned is not a casual omission. The document attempts to describe and interpret learning and understanding as present-tense phenomena of change whose nature is revealed in the changing, not in the amount of change as determined after the fact. In such a stance, data which illuminates learning or understanding as it was happening is valued much more highly than data which provides evidence of learning which has occurred previous to the data being generated. Data which is part of a personal transition is more likely to bring light to bear on the factors which interact with the process, and is more likely to sponsor our understandings of how such learning could be affected by changes in the interacting factors. Readers may anticipate that when content tests are used in descriptive and interpretive studies of student learning the test scores will be presented at least to provide triangulation for claims advanced in other ways, but in this study the tests were used only as part of the flow of instructional processes. All claims about the nature of what students learned or came to understand will be developed through the voices of more than a single student and through more than any one format, and each claim will be brought to light at more than one time during the study, to provide triangulation of a richer kind more suitable to the nature of the claims than could be provided by the mark-like data which the content tests generated.

The nature of the research method

> In place of attempts to subjugate research to a single, overarching theoretical scheme that is posited a priori, we might ... acknowledge that we [researchers], like teachers, cast around for ways of making sense of things as we address the situated problems of our practice (Cobb, 1994, p. 19).

> Some of the methodological implications of this view of mathematics learning [social constructivism] are still in the process of being developed, as the field attempts to craft a research approach that combines some of the desired features of laboratory studies of individual cognition with the ethnographic-sociological methods to which the classroom lends itself (Kieran, 1994a, p. 602).

> Research emanates from theory and leads to theory, sometimes through a long and circuitous route. Theory, however, is not a disembodied collection of abstract principles. Rather, theory is what results from someone's theorizing -- attempting to account for observations by building a body of coherent explanations and perspectives. At the same time, research is also a practical activity. Its aim is to inform our eyes and ears when observing and our hands and voices when acting (Research Advisory Committee of the NCTM, 1995, pp. 301-302).

> Attempting to do research through, on, or about one's teaching necessarily lands one in a complicated epistemological, practical, and intellectual bog. And the limitations and problems of such research deserve critical attention (Wilson, 1995, p. 21).

Educational theory cannot be tested directly for its effectiveness, because it must be translated into educational action before it can be said to make a difference to the learning of students. The instructional design of the strands of activity used in this study could be considered to be one such translation, in this case the enaction of a constructivist orientation to learning. Although I will develop this line of thought further in the next chapter, constructivism suggests that students must build their own understandings of concepts by interpreting their personal experiences through reflection and discourse. In some ways it would be fair to say that this study explored the relevance of constructivism to guide and inform both curriculum design and instruction on two levels. First, the students' mathematical activities were conceptualized in a constructivist frame, with small-group inquiry supplemented by multiple forms of classroom interactions. Second, as elements of students' orientations to learning, to mathematics, and to the role of student within the structures of school emerged as significant factors affecting their mathematical learning, the guided learning of more appropriate and complex orientations was developed from the same constructivist principles.

Although it was not the primary purpose of the research, readers interested in mathematics teaching might consider the possible applicability of the instructional design to other contexts. The strands of instruction described in the previous section could be considered separately or as a package as instructional practices for consideration by others. Alternatively, and more in line with the research orientation, the theory behind the strands of instruction could be considered to be the item being recommended for wider adoption. However, the research was not designed to provide evidence of cross-contextual applicability for the curriculum and instruction which was developed.

The particular curriculum was developed not to be disseminated but as a direct response to the pedagogic challenge to serve the particular students as well as possible. In terms of research intent, the role of the novel curriculum was to bring to light the personal elements which manifest themselves when students react to and interact with instruction that expects them to perform in unfamiliar ways. In this study the curriculum development was a means to achieve the study's pedagogic and research goals, not the goal itself. Similarly, the instructional decisions described in this study were enacted to support the students in their mathematical learning and in their learning about the factors which influence their mathematical learning. Because the intention of the curriculum design and the purposes of instructional decisions depart from a curriculum-validation focus, there are significant ramifications for the structure of the research and this document.

The first distinction can be explained as a resolution to an ethical dilemma which curriculum-validation approaches must face, especially when an interactive classroom stance as teacher-researcher is adopted. When generalizability is assumed or desired for research conclusions, it may be difficult for a teacher-researcher to deal with particularly personal elements of students' approaches to their work and their learning. Similarly, it may be difficult for a teacher-researcher to interact with students in interpersonal ways which may provide more support (or a different kind of support) for students in their learning than might reasonably be expected in other contexts. This study does not attempt to demonstrate generalizability or transferability of its specific processes or outcomes. Consequently it was not necessary at any time to consider pedagogic sacrifices to ensure that the curriculum as planned received a fair testing. Nor was it necessary to distinguish pedagogic responses to student-specific or context-dependent qualities actions from responses relevant to claims about generalizing a style of instruction. Rather, a teaching ethic defined any decision where data or generalizability concerns might override a student's educational opportunity. When a pedagogic effort brings to light a unique student's particular orientation to learning mathematics, that effort may provide the student with an opportunity to consider and perhaps change that orientation. The more effective that pedagogic effort is, the more complete is the research opportunity to view the student's orientation and the changing of the adaptation which the student may initiate. In other words, a proactive interactive stance as a teacher, always considering the educational opportunities of the students, can support without compromise a research objective to perceive the learning processes of the students as clearly as possible.

The "homework" system is a significant example of this sense of the relationship between instruction and research in such a context. As well as eventually providing significant research data, the homework discourse provided a significant flow of instructional feedback for me as teacher and it served as a significant process by which I could accomplish teaching responsibilities of building the instructional relationship and supporting, encouraging, and provoking students as they learned. Yet it was clearly more time-consuming than would be possible for teachers with multiple classes or multiple courses to teach. With the purpose of the research being primarily to explore and affect students' orientations rather than to test a process for teaching mathematics, the practicality for teachers of this kind of written exchange is not a concern in regard to the study design. Similarly, the student specificity and context sensitivity of such a process and its data enriches the research design and its findings rather than limits them.

Traditional curriculum-testing studies also require an operational definition of instructional success toward which instruction aims and from which the instructional design's value is determined. In many cases, this involves accepting the school's prevailing system of defining academic achievement, despite the problematic possibilities that such acceptance can entail. Assessment processes affect significantly students' understandings of what is valued, so using standard assessment processes may mean that students' sense of what counts as mathematical success will be reinforced by the assessment even when novel instructional processes provoke them to think otherwise. Also, reform-oriented instruction will likely misalign with a standard assessment process, skewing the results in favor of traditional instruction. Similarly, when assessment is designed to align with the research instruction, the scores of control groups with the same instrument may suffer from a misalignment with their instruction. This study is concerned with the nature of the learning and the factors which interact with the learning, not with measuring the amount learned for validating the curriculum design, and so its relationship with what counts as success for a learning sequence could be left more open, to be shaped by the teaching and learning as it occurred.

This can have a significant effect on the research's possible findings. When a research designer develops or adopts a definition of achievement and a normative operationalizing of that standard (such as a test and its resulting scores), what is to count as mathematical learning is defined by the research design, and cannot be viewed as a potentially problematic element of the study itself. Could what is normally taken to be mathematical learning and could the ways we normally tend to determine that it has occurred (tests) have significant effects on how students view their learning? More generally, should mathematical capability get pre-eminent status in terms of what the outcomes of mathematics instruction should be to the exclusion of elements such as personal confidence or students' understandings of what mathematics and learning are for? This study provided itself with room to respond to these questions as part of its inquiry into the learning of the students by not incorporating the use of test scores even to triangulate the data about student learning described earlier. To determine the factors affecting students' success in learning mathematics rather than the quantity of mathematical achievement sounds like a narrow distinction, but it implies the adoption of a present-tense sense of learning as something to be understood and supported by interacting with it while it occurs rather than something to be measured as an outcome. Testing when it occurred was a part of those interactions, not an independent measure of the effects of prior interactions.

The research method contributes in one more way to the nature of this document. The interpersonal teaching approach and the kinds of data gathering processes which were used necessarily result in data which is best interpreted within its contextuality: it is about the particular people who generate it. The data presented and interpreted in Chapter 1 provide a clear example. Benazhir is not made to represent a particular kind of learner; rather, Benazhir and her learning are shown as richly as possible without attempting to generate observations or conclusions which are necessarily valuable with others. When categories of learners are introduced, it is to help to interpret Benazhir's behaviors and learning, rather than to suggest that what Benazhir has done and said could be characteristic of other learners in other contexts. It is for the reader to make sense of Benazhir's learning in terms of its explanatory value in other contexts which the reader knows well, not the researcher-writer's task to suggest the imposition of truths about Benazhir's learning onto the learning of others in contexts which the researcher-writer does not know. It is the writer's responsibility, however, to purposefully and richly describe and interpret the writer's interactions with and understandings of Benazhir's learning, to support and encourage readers to trans-contextually connect from the writer's story to situations of the reader's choice.

This approach to writing is not simply a re-orienting of the research writer's relationship with readers of research reports. It aligns the writing with the style of the research and with the nature of the data. All forms of the data as outlined in the previous section of this chapter provided windows for viewing matters of significance to the people who were learning, and that learning was revealed to be a personal and interpersonal experience. It would be a denial of the nature of the data for this document to extract observations from the relationships which co-determined the deeply personal and interpersonal learning that occurred. Similarly, it would betray the personal and interpersonal nature of the conclusions to extract them from their context, stating them in a generic form that might imply they could be applied trans-contextually to other learners without due regard to the personal and interpersonal essence of teaching and learning.

When traditional research follows up its descriptions of purpose, method, data, and data interpretations with a section of general conclusions, those conclusions tend to be presented in a decontextualized form. Such conclusions can then be justified on the basis of the strength of their relationship to the data which the original context provided and on their demonstrated potential for resituating in other contexts. In other words, it may currently be more common for research reports authors to *extract* conclusions from their contexts by generalizing, as support for readers to *resituate* within the reader's own contexts the conclusions and/or the processes which led to them. In this document, each conclusion is presented as it arises during the data interpretation, supported and enriched by other data involving other learners. This provides support for the reader to judge the conclusion's justification both within the context in which it is presented and its relevance for any contexts to which the reader may which to transpose it. When conclusions are stated in ways which preserve their situated, personal, or interpersonal character, readers must find within the document sufficient support and encouragement to *trans-situate* or relocate any conclusion and/or process formed to another context. That support and encouragement must emerge for the reader within the rich descriptions and interpretations of the research story. This document attempts to recognize the relatively uncommon expectation which this writing stance may place on some readers. As an acknowledgement of the unsettled and rich diversity of perspectives on the relationships of conclusions with context in educational research, at the end of Chapter 8 readers will find the conclusions developed earlier in the document within particular contexts stated more traditionally. If they appear somewhat naked in that form, readers are encouraged to revert to the all-dressed versions of each conclusion developed at earlier points in the document.

This document's research stance about what counts as data, about how data and conclusions should be related, and about how research documents should present conclusions is part of an ongoing proactive search for how research should be conducted, interpreted and shared with others (Cooney, 1994; Graavemeier, 1994; Kieran, 1994a). Comparisons and distinctions will be drawn with three sets of research as a way to clarify further the research stance and methods used in this study.

A similar shift in stance in regard to what counts as data can be inferred from the writing of Cobb and associates about an extended research study with similar goals to this study's broad purpose (Yackel, Cobb, Wood, Wheatley, & Merkel, 1990; Cobb, Yackel, & Wood, 1991; Cobb, Wood, & Yackel, 1991; Cobb, 1995b; Yackel & Cobb, 1996). When that study began, a grade 2 classroom was taught for a full year in ways compatible with a constructivist orientation. To validate the orientation of their pedagogy for different audiences, they tested the students' exit understandings using different tests with different orientations. As the study has progressed, it has involved more classrooms and has expanded to other elementary grades. Concurrently, the focus of what counts as relevant data has shifted more to classroom intercourse and students' writing. This data better illuminates the factors which are significant to the successful adoption of more complex sociomathematical norms, a central element in the negotiation of more effective student perspectives on mathematics and on learning.

Cobb and associates are some of the few researchers in mathematics education who are incorporating changing students' orientations into their instructional design as well as their research design. As with this study, although the target of attention for Cobb and his associates is well-defined in the research design, the nature of the success of the instruction is left to emerge from the particular interactions with the particular students rather than be pre-defined by the operationalization of an assessment procedure. With this study situated in grade nine, the relationship of the students with mathematics is even more tightly interwoven with their relationship with the authority of the school in assessing their mathematical achievement. Also, this study is designed to be more sensitive to differences among learners and the learning that they do than are the studies of Cobb and associates. As a result, the shift to more open and context-interdependent data sources than testing becomes even more essential than in the Cobb studies.

Romagnano (1994) also reported on a study in which a reform-oriented curriculum was offered, and he also reported that what students thought counted as mathematics was of significant importance. The inquiry-oriented problem-solving activities which he and a teaching colleague provided to their junior high school students are similar in general design to the inquiry activities in the Patterns strand of this research. Romagnano found that the students' roles needed to be very different with this kind of learning than with traditional instruction but that the students were persistent in their traditional stances. Also, students' traditional expectations of teachers to make procedures clear and to sequentially and directly lead them through activity was incompatible with inquiry instruction. Romagnano focused more on the difficulty of teaching through inquiry methods in the face of these incompatibilities than on the factors which could contribute to supporting students as they change their sense of mathematics and of their roles as learners. As the incompatibilities mounted, Romagnano also found that the inquiry instruction did not provide a natural fit with the school's (and the students') expectations regarding how evaluation of students would proceed. Unfortunately, all the mismatches became reasons for the new curriculum to fail despite the author's certainty that it was better mathematics and better pedagogy.

In this study, those same facets of student orientations which were barriers for Romagnano were targets for the curriculum along with the mathematical content. Students were encouraged and directed to inquire into what mathematics is, how they learn, and how they relate to the authority of school and its evaluation procedures, all concurrent with and intrinsically interwoven with the mathematical aspects of their inquiry. At the same time these aspects of students' orientations to learning mathematics were targets for the research.

How could instructional decisions best foster students' consideration and changing of their epistemology and ontology in relation to the learning of mathematics, and how would those changes be affected by and affect the ongoing learning of mathematics? For example, in Romagnano's study the difficulty of aligning assessment processes with inquiry-style learning was compounded with the necessity to provide assessment which the students felt was appropriate and fair in the given context of the school's assessment patterns. Romagnano could only conclude that schools and students must change if inquiry-based instruction is going to be as successful as it could be. In this study, the challenge of arriving at suitable assessment procedures was posed by me openly with the students, as an inquiry that I had to explore, discuss, and solve with their significant involvement, much like they proceeded with the mathematics of the Handshakes inquiry. The fact that assessment was not pre-defined by the research design's data needs or by the school's specified evaluation system meant that the study could explore the significant role of student evaluation as an influence on (and as influenced by) students' mathematical, epistemological, and ontological learning.

Inquiry-oriented learning was also at the heart of a study by Borasi (1992). She also found though prior experience in instructing a class of students that students' orientations to their roles as students had to evolve to include a different understanding of mathematics and of learning. Borasi discusses the importance of evaluation which is philosophically in synchrony with inquiry as a learning process. "My double role of researcher and instructor provided considerable advantages. Since I was designing the very unit I was teaching, I had many opportunities to adapt the curriculum to fit both my research goals and the specific needs and personalities of my students" (Borasi, 1992, p. 9). However her study does not explore or describe ways to help students with traditional perspectives on being evaluated to perceive and incorporate into their own values the values behind the evaluation processes being used. Borasi's study was a valuable example for me in its dedication to interpreting personal student data from within a pedagogically intimate teacher- student relationship. However, in part because of limitations determined by the particular school context (an alternative school) and in part because her most intimate data flow was from within a particular special-case instructional relationship, the Borasi investigation could not explore the interaction of classroom-level evaluation processes with each student's learning of mathematics and with their epistemological and ontological learning to the extent that this one does.

The role of academic voices

I would like to make explicit my reasons for providing blocks of quotes from the academic community in the openings of chapters and sections without explicit links from the quotations to the rest of the text. The situating of quotations in such a way was done to help readers privilege the voices of the students on which this study depends rather than the voices of the disciplines, while at the same time preserving relationships between this study and voices of the academic community. This research and the document itself were developed within a personal relationship with the writings of many other researchers, educators, and theorists, and their inclusion is important as a matter of principle. Of course, carefully selected quotations provide readers with links to other academic voices and invite readers to apply particular theoretical elements to their interpretations of the study, especially when each quote's selection and insertion is explicitly explained by the writer. Both these goals can be achieved without the author making each intended purpose explicit, leaving readers free to decide for themselves the relevance of each quotation and each group of quotations.

Academic voices tend to make strong claims dependent on complex relationships among abstract ideas and terminology for those ideas. This is what lends them their power and value in academic writing, and in the case of this study this means that they provide a valuable balance for the context-interdependent nature of much of the data and its interpretation. Yet that power can also tip the tone of a document from other purposes toward issues surrounding the words and/or the abstract ideas themselves. The discourse of educational theory and research tends to present theory as distinct from context and action, a separation which is not representative of the design or intention of this research. Extensive use of and explication of academic quotes could suggest a privileging of abstracted or generalized conclusions over conclusions which maintain an intimate relationship with the context in which they were made. Such a privileging would be antithetical to the claims of this document. More significantly, the telling of any story cannot provide room for rich interpretations of all the voices that are interwoven into the tale. If this story is to be woven primarily within the context of the classroom from the actions and voices of learners and their teacher, then the privileging of academic voices and the adoption of the tone that accompanies them is problematic. For a research account to maintain the focus on the flow of events and the students' interpretations of those events, academic theory must be held in check lest those less verbally adroit voices be overwhelmed.

Will readers view the selecting and placing of quotations as haphazard or meaningless simply because their intention for the writer is not specified? I hope not. Although I cannot prevent it given the stylistic choice I have made, I would not even be happy if readers viewed them as one might with salt and pepper, to be sprinkled over the main course according to taste. For me, they are a central part of the recipe, but my choice not to let them overpower the other ingredients, especially the fragile voices of the students in the other chapters, required me to explore other options for their use. As a result, throughout the chapters, readers will encounter quotations placed in such a way that they may flavor the readers' interpretations of the section or chapter which follows, without necessarily making the contents of the quotations the focal element or making the students' words compete with them for attention. In most cases, the group of quotes works to suggest a particular frame for interpreting a particular section of text, but occasionally, most notably at the end of Chapter 6 and the beginning of Chapter 9, the quotations provide a tantalizing element of diversity within the theme at issue, before the flow of events and students' words provides a frame for viewing the theme in context.

The chapters

> Thoughts must be opened into sequential prose. It would not do to lay them out too precisely, however, for I have wanted to convey something of the process of learning, and most learning is not linear (Bateson, 1994, p. 30).

> In developmental research, making sense of what is going on is more important than prediction. Here the experimental experiences are subjects of an interpretive process. The researcher tries to make sense of what is going on in the classroom against the background of the thought experiments that preceded the instructional activities (Graavemeijer, 1994, p. 454).

> Meaning may be conveyed through narratives of experience which, if translated into categories, will lose crucial dimensions. Meaning is further lost if the 'knower-subject' is acted upon as a spectator or object, separate from personal meaning. Objective measures treat the "knower-subject" as object or spectator, and attempt to "abstract content from the human experiential realm and fasten it to a grid of formal logic, thus creating a diagram of a flat plan of objective reality external to human experience' (Polkinghorne, 1988, p. 135). (Campbell,1995, p. 449).

> The purpose of such interpretive research is not to determine whether general propositions about learning or teaching are true or false, but to further our understanding of the character of these particular kinds of human activity. It is a narrative rather than a logico-scientific tradition of scholarship (Lampert, 1991, p. 123).

> This coming to consciousness occurred through a unique method -- what might be called simply telling stories... storying and restorying experience, a mode of narrative inquiry about one's own experience (Lyons, 1995, p. 75).

The first and last chapters serve as bookends for the document as a whole. Benazhir is not prominent in the intervening chapters before the last chapter brings her back into the foreground. In the interim, the words and actions of other learners in the classroom provide the focus for the reader's attention. Each chapter from Chapter 4 to Chapter 8 describes a particular aspect of the algebra instruction and follows particular students as they experienced the tasks and made sense of the resulting experiences.

Chapter 3 sets the stage for all the chapters which follow by describing the development of the instruction on which the other chapters report. It offers the theoretical background for the instructional strands described earlier in this chapter, and it offers a way to make sense of the learning as it unfolds. As mentioned earlier, Chapter 3 explains in detail what is meant by a constructivist orientation, referring to the events of Chapter 1 for examples. The chapter explains how constructivism supports interpretation of students' learning actions whether they are learning about mathematical concepts, about how they learn, or about how they want to interact with school structures as students. It is the only chapter other than this one which does not build upon student voices.

Chapters 4 through 7 return to the telling of the events of the instruction and research. Chapters 4 and 5 report on the design and implementation of a task for teaching the students about algebraic variables and the arithmetic of algebraic expressions. Through selected portions of transcripts from recorded conversations, Chapter 4 introduces a group of four learners while they engage successfully in the inquiry. However, Chapter 5 plots a parallel course with two other groups whose success with the inquiry was very different. In one sense, this is a turning point in the flow of the document. Like the first two chapters, these two chapters are intimately connected to the teaching and learning of mathematics in terms of constructivist pedagogy. However, these chapters incorporate as a concurrent theme the complications which affect students' learning of the content. Students are shown to have different senses of their role as students, with different senses of how they should proceed with the learning opportunities provided. They have differing expectations of the teacher, and they position themselves in very different ways in response to the role of evaluation and marks in math class specifically and school in general. All of these differences are shown to be relevant to the learning of mathematics. As the story unfolds through further chapters, the learning of mathematics diminishes in prominence as the focus shifts to the students' concurrent learning about epistemological, ontological, and identity issues.

Chapter 6 is concerned with students' learning about learning. Its data is primarily the written discourse which was the homework for each inquiry-based lesson. This data provides a more personal and more chronological view of students' orientations to learning mathematics. Three students from the same learning group are shown to have very different epistemologies. However, they are also shown to have engaged differently over the course of the inquiry in reflective thinking about their understanding of understanding and how it develops. As a consequence, despite learning together for the full course of the inquiry, the more complex understandings of these personal concepts which each of the three students developed during the course of the inquiry are significantly different. Chapter 6 closes by claiming that when epistemological stances change in ways that allow students to learn more effectively, then they have learned epistemologically. When that change is in part attributable to teacher activity which intended to bring about the change, then epistemology has been taught concurrently with the mathematical instruction that is the foreground of the lessons.

Chapter 7 is a long chapter. First it describes the resolution at the classroom level of the issues and difficulties raised in Chapter 5. This leads into a description of the negotiating of assessment processes with the class, and the learning that emerged from that process. The homework discourse of three students provides a window on the range of orientations toward assessment issues for mathematics students and the ways those orientations influence their relationships with the mathematics, the teacher, and the school in general. The chapter suggests that these issues are issues of identity for students, interdependent with their ontological sense of themselves as learners generally and as students within the authority structures of the school and the mathematics classroom. The writings of the three students are followed as they engage in the assessments which were part of the inquiry program's instruction, and the changes in their ontological understandings are chronicled.

Chapter 8 is a concluding chapter in more than one way. Its data is from the closure conversations described earlier in this chapter as the final interaction between the students and me. Its focus is on the group of students who were shown in Chapter 5 to be most uncomfortable with the difficulty of understanding the mathematics and adjusting to inquiry as a style of learning. As they describe their resolution of the concerns that were so significant then, they describe the whole research program retrospectively. As they look forward to the mathematics they will study in high school they portray themselves as having changed significantly in terms of their expectations of themselves, of teachers, and of school authority structures generally. The chapter closes with general versions of conclusions attributable to the study as a whole, each of which have been stated in previous chapters within the context of specific instructional processes with specific learners.

Chapter 9 offers a final image of the particular instructional relationship which was featured in Chapter 1. It chronicles one aspect of the ongoing learning which I as a researcher-instructor was led to begin during this research. On one hand, it portrays the significant limits of a teacher's efficacy even when that teaching is positioned within interpersonal discourse with students. On the other hand, it describes the opportunity of the teacher who is positioned in such a way to learn about himself as a result of those interactions.

If the reader is progressing in order through the chapters, it is now time to read Chapter 3. It begins by discussing as a curriculum design process the algebra tiles activities that Lorna, Rose, and Benazhir were doing in Chapter 1. It then provides an elaboration on the ways in which a constructivist epistemology can be developed to be relevant to the planning, enacting, and interpretation of instruction to help students learn mathematics and, concurrently, to learn about learning and themselves.

Chapter 3. Is This Constructivism?

Algebra tiles and reforming algebra instruction

I first encountered algebra tiles as a teaching idea twenty years ago. Yet, their use is still viewed as an innovation and has not been widely adopted for use. "The general area of school algebra pre-16 shows relatively few variations. ... Although there is considerable agreement on how to teach algebra, comparative studies suggest that the task is not very successfully accomplished. There is still a marked need to improve our teaching" (Howson, 1991, p. 20). When algebra tiles can provide a context in which students can develop a rich understanding of polynomial arithmetic, why has their use been so slow to spread?

The slowness can be attributed to a variety of possible factors. I will describe four of particular relevance to this study. One, incorporating algebra tiles into an algebra unit is not straightforward, and teachers have not known what to do with them: what should be taught by direct instruction, what should be done with the terminology of polynomials, how much practice of what kind should students do, and how and for what would the teacher test ? Two, their use tends not to fit directly into the social contract teachers and students are used to following. This contract involves being shown an arithmetic procedure, practicing it for automaticity, and being tested for reproducing the process with questions similar to the practice exercises. Traditional instruction with various types of polynomials to be factored can easily be organized within this standard contract. Three, the value of algebra tiles is not clear to those who view mathematics as answer-getting: why bother to learn an answer-getting process which relies on a kit, one that you will not likely use after learning with it? The value of having students understand operations which generate answers and having students connect those understandings with other understandings is not recognized widely among teachers.

> Education that emphasizes memorization and drill and neglects understanding may account for many of the well-publicized failures of American students. ... that a simplified, easily master curriculum actually does a disservice to students by limiting opportunities for real engagement is a tenet underlying much recent research in education (Lewis, Schaps, & Watson, 1995, p. 549).

Four, there are many teachers who view algebra's situation as a "critical filter" (Wagner & Kieran, 1989) as a positive aspect of the current situation. "Because of algebra's 'gatekeeping' role, the study of its underlying ideas in all mathematics classes would be an important element of any reform program that purports to provide all students with opportunities to learn real mathematics" (Romagnano, 1994, p. 11. See also Lewis, 1989, p. 248.) However, if one believes that there is nothing wrong with some or many students not learning as well as others, since only a proportion of students should progress further in math, then there is no reason to improve or alter that instruction. Those who find it easy to learn to manipulate algebraic expressions go on, and those who don't, don't. It may not be unfair to compare such an orientation to a track official refusing to lower the hurdles just because it would make it easier for more participants to do well. If it is a competition's primary purpose to eliminate the less capable athlete or to challenge the exceptional athlete, lowering the hurdles doesn't make sense. It is only when we leave the competition metaphor and frame the conversation in educational terms that it makes obvious sense to have as many students as possible learn as much algebra as richly as they can.

One significant difference in high school mathematics is developing, however, which may mean that mathematics pedagogy must change: algebra is no longer a subject for a select few. At a time when the proportion of students graduating is itself increasing, "More [U. S.] high school graduates are taking three years of mathematics -- from 37 percent in 1982 to 60 percent in 1994" ("Report on mathematics," 1995, p. 5). Enrollment in algebra 1 has climbed from 65% in 1982 to 81% (and as high as 95% in some states) in 1990 ("Report reveals increase," 1991). In algebra 2, enrollment increased from 37% to 56% from 1982 to 1992 ("Study reports more mathematics," 1994). As more and more students attempt and expect to successfully pass through the critical filter which is high school algebra, it is difficult to imagine that the filter itself will not change.

What would algebra tile use mean within a traditional approach to teaching algebra? Certainly teaching with algebra tiles is not more efficient than direct instruction of factoring. "It takes time to *do* things, to use them in order to know them. Purveying information takes a fraction of that time" (Sizer, 1984, p. 95). Nor does it necessarily provide more students better facility with factoring skills. The provision of a context within which the arithmetic of algebra can be seen to make sense does not necessarily mean that students get factoring questions right more often. In fact, students who perceive the learning of mathematics as being the memorizing of arithmetic procedures are unlikely to view the added responsibility of learning how to factor with algebra tiles as valuable, declaring as unproductive the time used in processes such as noticing patterns in how and why algebra works. Although algebra tiles may provide opportunities for teachers to address such perceived limits to students' responsibilities, clearly their possible pedagogic value within the algebra curriculum lies outside the concerns of traditional instruction, with the depth of understanding of algebraic expressions generally and with factoring and multiplying in particular. "A mismatch exists between the pedagogy of current reform and the basis on which mathematics teachers have traditionally felt efficacious in directing student learning (Smith, 1996, p. 387).

Using a manipulative such as algebra tiles does not ensure a shift in learning from reception and practice to construction of meanings from experience. For instance, I feel that my use of algebra tiles early in my teaching career was still within a traditional instructional frame, in that I showed by example how each kind of expression could be portrayed with the algebra tiles and provided a set of exercises which the students did with their algebra tiles. Next, I provided direct instruction describing the general-case factoring procedure and illustrative examples for each kind of polynomial (for instance: binomial common factor, trinomial by inspection, trinomial perfect square). I ensured that the students wrote the procedure down and labeled it appropriately, and then required the students to do practice questions without the kits based on the principles we had 'derived' from the questions done with the kit. The students' view of legitimacy for the kit depended quite simply on the inclusion of test questions such as "Draw a rectangle showing the factors for $x^2 + 5x + 4$." The use of the kit to view polynomials as rectangles and factors as dimensions had become more content. A particular meaning for factoring, based on multiplication as an area idea, was taught, but nothing else changed. It was still direct instruction, with my sequencing the students' interaction with knowledge and breaking it down into its composite facts (notes) and skills (practice questions).

The application of constructivism to mathematics classes is neither easy (Brown, 1994; Driver & Scott, 1995) nor widespread. "Put concretely, inappropriate pedagogy affects pre-algebra and algebra classes as much as it does remedial and regular eighth grade math classes. In the 1986 NAEP study, *neither* upper nor lower quartile 13-year-olds reported much use of constructivist, student-centred approaches" (Stedman, 1994, p. 30). Change in mathematics education moves slowly. "We continue to offer a curriculum consisting of eight years of eighteenth-century arithmetic, three years of seventeenth-century algebra, and one year of third-century B. C. geometry" (Romberg, in Parker, 1993, p. 6).

As direct instruction and assessment narrowed what comprised algebra's core to a particular set of performance skill outcomes related to factoring, those educational outcomes developed a hollowness to the point where some mathematics educators considered the algebra subtopic completely empty of value (Usiskin, 1980/1995). Sizer calls the products of such goal-specific teaching of algebra, "competent algebraic drones" (1984, p. 188). Even as algebra's place in high school mathematics was first being secured, its calculation procedures were not seen as particularly pragmatic: "We should not care much about training algebraic computers; the 'practical' utility of even the simplest algebraic computations, as such, is not widespread" (Thorndike et al, 1923, p. 328). Yet, algebra maintains its place at the centre of mathematics at the entry to high school mathematics, perhaps literally defining that entry: "One out of four [U. S. high school] students never takes algebra, ... and half the students who do take first-year algebra leave the course with a lifelong distaste for mathematics" (Steen, 1992, p. 258).

The orientation toward learning algebra which was portrayed in Chapter 1 has the power to address issues of accessibility and success for students. Addressing such issues is dependent on pursuing potential mathematical outcomes of learning which lie beyond the narrow band of outcomes measurable by a test of factoring questions, with or without algebra tiles.

> Much of the power of mathematics comes from its extensive use of symbolic notation -- not only to identify objects, but to describe operations, and moreover, not only concisely but *suggestively*, for it is in the way in which symbolism can lead us to new results that its power lies. Yet this great use of symbolism is double edged: it gives mathematics its power, but it also makes the subject difficult to learn and to teach (Howson & Mellin-Olsen, 1986, p. 11, italics in original).

Students must learn about algebra in ways that make the power of its symbolism accessible to them, rather than in ways that use that power against them. "The meaning of a symbol is an intention to act" (Steffe, 1995, p. 87).

When I watch students re-view their understandings of numerical arithmetic as they learn how variable arithmetic works (as Benazhir did), or when I watch students develop and express a complex awareness of the uses of variables and algebraic expressions to describe mathematical objects and the actions they take on them (as Lorna did), I have no difficulty suggesting that learning factoring with algebra tiles has value for school mathematics. This claim could quickly lead to many research questions from within the teaching and learning of algebra content , such as how best to use algebra tiles and what can be accomplished with them are two. That is not the direction which this study takes. In this work, I intend to use the algebra tiles pedagogy as a single element in a broader look inclusive of, but not limited to, curriculum and instruction questions.

> These [mathematics education] reforms are visions, not programs for practice. They ... demand many changes: in standards, curriculum, assessments, and instruction. But underlying them are changes more fundamental still: different views of knowledge and different ideas about the nature , purpose, and scope of school subjects. The reform instruments represent new conceptions of learning and a serious commitment to serve a diverse student population well. They also entail new images of good teaching (Wilson, Peterson, Ball, & Cohen, 1996, p. 469).

This study is an exploration of the multiple facets that affect and are affected by the reform of mathematics education. Chapter 1 is an example with particular emphases among the factors that are at play.

I would like to claim that in Chapter 1 evidence is given of students learning more than just algebra. They are learning about what mathematics is, what it means to learn mathematics, and (separately) what it means to be mathematics students. This broadening of scope will lead in later chapters to dealing with what students should learn within algebra and about algebra; with what students must learn about learning mathematics, and with what teachers and students must learn about how teachers and students and mathematics interact in school. My intention is to present the various elements I have named as interdependent aspects of a single dynamic system, none of which can truly be understood or changed independently of the others.

As in Chapter 1, I hope that throughout this document the learners' voices provide the weft in the tapestry which each reader weaves. At times, particular students will be heard at length, while at other times the voices of many students will provide a sense of range in regard to how they are learning. However, a tapestry cannot be made from weft alone. What yarn can I begin with, to structure the tapestry which Chapter 1 begins to relate? I have chosen to begin with a particular question which has the curriculum issues and the students' learning at its core: what do I mean, when I describe the teaching in Chapter 1 and the rest of the algebra instruction I provided as *constructivist*? More practically, how can a constructivist epistemology contribute to directing mathematics pedagogy when the scope of that pedagogy is expanded as outlined above? I will begin to answer these questions by elaborating on a series of claims about constructivism.

Constructivism is a family of epistemological stances which implies uncertainty in teaching.

There has been some discussion about whether "constructivist" is an adjective appropriate for teaching or pedagogy (Cobb, 1995c; Hoyles, 1988; Lochhead, 1991; Orton, 1995; Steffe & Kieren, 1994), as its primary use is to label a family of epistemological stances (Bereiter, 1994; Cobb, 1994; Noddings, 1990; Phillips, 1995). Generally, it does not seem odd that a significant shift in how we think knowledge or understanding comes into being would inform what educators do. However, even if a claim could be made that it is illogical to talk of constructivist pedagogy, it has become a common usage. Constructivism suggests that learners must be active, probably active in a physical sense, but certainly active in a mental sense. In the following quotes, the active verbs suggest the complexity attributed to that student activity.

> Individuals construct their own reality through actions and reflections on actions (Steffe & Kieren, 1994, p. 721).
>
> To 'do' mathematics is to conjecture -- to invent and extend ideas about mathematical objects -- and to test, debate, and revise or replace those ideas (Schifter, 1996, p. 495).
>
> Individuals organize their experience in their own subjective ways (von Glasersfeld, 1991a, p. xiv).
>
> Knowledge emerges only through invention and re-invention, through the restless, impatient, continuing, hopeful inquiry men [and women] pursue in the world, with the world, and with each other (Freire, 1970, p. 58).
>
> To learn mathematics successfully, students must construct their own understandings, examine, represent, solve, transform, apply, prove, and communicate. This happens most effectively when students work in groups to discuss, make presentations, invent, impose their interpretation on what is presented, and create theories that make sense to them, thinking critically and in terms of relationships (Adams & Hamm, 1994, p. 188).

> Children learn through exploration. They construct their knowledge through direct experience (Tyler, in Hiatt, 1994, p. 787).

> Learning [is] a generative process of meaning-making that is personally constructed, informed by context and purpose of the learning activity itself, and enhanced by social interactions (Borasi, 1992, p. 3).

> Children *reorganize* and *reconstruct* experiences of their physical and social environment (Driver and Scott, 1995, p. 28).

> Mathematics can be built up only brick by brick (Peter, 1961, p. xiii).

The verbs in these constructivists' statements do not imply simple actions by the learners. Because of such complexity, pedagogy with constructivist goals cannot view curriculum and instruction as well-defined and predictable. "Teachers who begin this process expecting to develop a finished repertoire of behaviors that, once achieved, will become routine will be disappointed. Teaching this way is necessarily disruptive of routine, if for no other reason than that the students will continually surprise us with their own discoveries" (Schifter, 1996, p. 499). The involvement of students in complex ways depends on and makes relevant each student's perspective on learning and on school and their view of what counts as mathematics. Also, many of the actions which constructivists hold to be central to learning require interpersonal activity, making the social nature of the particular classroom and particular students relevant. This relevance means that the learning that occurs is unlikely to take place in ways simple enough or straightforward enough to be viewed as predictable and manageable outcomes of a curriculum design or a teacher's choices. Whether these choices are in the planning or the enacting of the lesson, ultimately the students must be active in deeply personal and interpersonal ways and these cannot be considered fully predictable or manageable by the teacher.

> Perhaps most importantly, developing specific strategies in the abstract violates the concept of connecting with students. ... In order to connect learning to individual experience, educators have to determine what confirms and contradicts particular students in particular contexts (Baxter Magolda, 1992, p. 225).

I am not suggesting that I had little effect on the learning of the three students from Chapter 1 or that it was a serendipitous and spontaneous outcome. Nor am I suggesting that the choices I made (in the design of the sheets, their implementation, and my interactions with the students while they learned with the sheets) where insignificant. Instead, I want to claim that a constructivist orientation to teaching, because it recognizes and (hopefully) amplifies the role of each student's uniqueness in their learning, must admit to considerable uncertainty regarding the cause-effect relationship between teaching and learning. A teacher can influence the learning outcomes but cannot predetermine them.

Mapping out constructivist teaching

What then does the teacher do if we attribute so much of the acts of learning to the students? This section describes a progression from a teacher's task to students' activity, and from student activity to the deriving of meaning from personal experience.

I think it is useful for teachers to consider their starting point as designing or selecting student *tasks*, rather than thinking of curriculum as content first. The original guiding question can be, What can students do that will include their encountering the targeted mathematics in their actions? Addressing such a question is very different from considering what knowledge the students should receive. As Dewey warns,

> When education, under the influence of a scholastic conception of knowledge which ignores everything but scientifically formulated facts and truths, fails to recognize that primary or initial subject matter always exists as a matter of active doing, the subject matter of instruction is isolated from the needs and purposes of the learner, and so becomes a something to be memorized and reproduced upon demand. ... Only in education, never in the life of farmer, sailor, merchant, physician, or laboratory experimenter, does knowledge mean primarily a store of information aloof from doing (Dewey, 1916, quoted in Sizer, 1984, pp. 94-95).

This claim guides what the teacher might design or select as a student task but it does not prescribe how that task will become actual *activity*. "Good tasks are easy neither to design nor to replicate because they depend on specific students and on the classroom environment in which they evolve." (Reys & Long, 1995, p. 296). As teacher and students in a particular classroom interact, a teacher's designed task becomes the students' activity, deeply influenced by the particularities of the students and the context. The teacher can negotiate with the students the flow of the activity indirectly by influencing the context and directly by assisting and guiding the students.

At this point it is useful to distinguish between the activities of the students and the experiences of the students. The students, collectively with the teacher and individually, will determine the nature of the experience by the sense they have of their activity. Even then the *experience* is not the *meaning*. We can call the meaning of a lesson for a student the derived understanding of her experience. As students change activity into personal experience with the interactive presence of the teacher and their fellow students, and then obtain personal meaning for that experience by understanding it, again with the interactive presence of all those in the social situation, we can say that learning is occurring and therefore teaching is occurring.

One way to understand instances one has experienced is to reflectively generalize those experiences. "Mathematics, like any other subject, must begin with experience, but progress toward abstraction and understanding requires precisely that there be a weaning away from the obviousness of superficial experience" (Bruner, 1979, p. 121). A particular experience is no longer superficial when it is situated by the learner with other experiences and both similarities and elements of uniqueness are signified. Experience achieves its full meaning in the *discourse* that is simultaneously about the experience and part of it.

In this regard, it is important to keep in mind that discourse could well have been part of the activity itself, prior to whatever discourse is devoted to making sense reflectively of the experience. Further, the reflective discourse itself can be a part of the experience, subject in a recursive sense to being the subject of reflective discourse. A simple example of this recursion brought explicitly into view will be explicated in a further chapter, based on my asking students the homework question, "What is the purpose of homework questions like this one?" Discourse can be situated as a part of the activity or as a part of the experience, as well as being the process by which meaning is derived from experience. This is not a simple answer to "a theoretical issue of great importance to current constructivists: the priority that one ought to assign to *experience* versus *language* in accounting for knowledge acquisition" (Prawat, 1995, p. 14). However, it is an answer that allows the interpretation of students' acts of understanding of mathematics and their acts of understanding their own understanding.

Because understanding is so dependent on factors which the students bring forward, the choices which a teacher makes cannot be said to determine the learning which occurs. Even as the teacher interacts with these factors, it is the students' actions and discourse which are most significant in terms of the particular learning outcomes achieved. Thus, constructivist teaching must accept a non-deterministic relationship with its desired outcome. The teacher can plan the task. The teacher can interact with the students in significant ways as they make the task into activity. The teacher can be an integral part of the discourse by which meaning is derived from experience. However, the teacher cannot cause the learning. Yet, this is only a caution against expecting predictable outcomes, not in any sense permission for constructivist instruction to be laissez-faire. "Although subject-centered instruction overestimates students' receptivity to canned instruction, activity-oriented instruction underestimates their need for adult guidance, trusting too much the child's own innate capacity to organize or make sense out of individual experience" (Prawat, 1995, p. 15. See also Adams & Hamm, 1994; Wells, 1995.)

Consider the pedagogy of the algebra tiles described in Chapter 1. The tiles and the sheets were the backbone of the task that I as teacher had planned. Influencing the students' sense of the task's goal is part of the teacher's framing of the task for the students. In this case, the intention of the task was the exploration of the ideas of factoring and algebraic notation as they are related to rectangles made with the kit. The framing of the task included setting structural aspects such as the student groups and influencing mediatory aspects such as students' support for each other and students' pace with the activity. Within that framing, the students enacted the activity, making and sketching rectangles, recording the areas and dimensions of the rectangles algebraically, and performing the mental activity that accompanied these visible acts. The students' enacting of the task can be considered experiences to the extent that the students made sense of what they were doing. The students and the teacher mediated these experiences socially, expressing and negotiating meaning, simultaneously compounding the experience with the experience which was the social discourse. As the teacher was observing and actively participating in the students' learning, the teacher was able to construct an understanding of that learning. Overall, the students' activity with algebra tiles as described in Chapter 1 could be considered a classic example of constructivist learning. Looking more closely at the instructional decisions that sponsored that activity can provide details about what is involved in framing instruction from a constructivist perspective.

Constructivism means more than physical and verbal activity.

I do not mean to suggest by the particular lesson described in Chapter 1 that constructivist pedagogy requires the use of manipulatives. From a research perspective, manipulatives offer a significant extension of the possibilities for investigating cognition: when the students are using manipulatives, the observer has access to visual evidence of student activity in making and amending meaningful images, as well as the aural or hardcopy evidence available when students express the images and meanings verbally, pictorially, and symbolically. On the other hand, as curriculum designers we cannot fixate on the objects themselves as causing the learning that occurs. Marilyn Harrison said half of this, when she said, "The mathematics is not in the manipulatives, but in what the students do with the manipulatives" (Skemp, 1990, personal conversation). This is only true if we think of what the students do as much more than what they can be observed to do. Mathematics educators must interpret broadly the idea of "doing mathematics," in which not only the physical activity, but more particularly the mental and interactive activity, comprise the mathematical activity. "Learning mathematics by doing it means not just handling objects, but also thinking about the handling, and reflecting on the processes and products" (Bishop & Goffree, 1986, p. 318).

What are the implications for pedagogic practice of this claim? Regardless of the quality of the activity, and regardless of whether that activity is primarily physical or tactile or verbal or symbolic, its value lies in the meanings that the students construct as they engage in understanding it. Through classroom verbal interaction each student's meanings for their experiences (including the interactions as experiences) are subject to the mediation of peers and the teacher, as their attention shifts from what they did to what they think about what they did (d'Ambrosio, 1995; Freire & Macedo, 1995; Gutierrez, Rymes, & Larson, 1995; Morrow & Morrow, 1993; Prawat, 1995; Wells, 1995). I do not want to claim that discourse is necessarily separate from activity and experience. Discourse and sense-making in general must be ongoing during activity, and discourse itself is an experience which should be subject to reflective sense-making. Regardless, students must be guided to actively make sense of their experiences. Even in a traditional approach to the arithmetic of algebra, for instance, such a need for guidance in sense-making would strongly suggest a discussion following an assigned exercise, pursuing the common elements within a set of exercises and how those elements contributed to the students' determination of answers. "Without reflection, practice can wash over you, leaving no permanent marks" (J. Mason, Burton, & Stacey, 1985, p. 149).

The importance of guiding students to make sense of their activity explains the particular structure of the algebra tiles sheets in Appendix B. Sets of traditional-looking "practice" questions provided chances to explore rectangle imagery and relate it to arithmetic understandings. Following each set of practice questions were less prescriptive questions directing the students to do something on the back of the sheets. Responding to these questions meant in some way expressing the images and relationships that characterize each set of questions and their responses. The instructional purpose of the practice questions expects that each student will recognize and express the commonalities of the experiences triggered by the individual questions. Some students have always generalized from individual practice questions as a matter of course during traditional mathematics instruction while others have failed to recognize the importance of doing so. For all the students, the open questions were a chance to engage in noticing and expressing patterns and to develop a sense of the importance of doing so. It is not that the students left their manipulatives behind when they did these questions: often they used them extensively. Nor did the students necessarily express their image of the relationships by using the forms of traditional algebra. However, they were being directed to express their sense of the patterns and connections which they were noticing. For students who were not noticing patterns or interpreting them, these questions served as explicit invitations to engage in the complex learning that is expected within constructivist teaching.

> Abstraction and abstract reasoning develop with time, constructed from experience. ... The child has to have experienced something potentially mathematical to recall, visualize and talk about. The content has to be real to the child. Put differently, it is not possible to talk mathematics if the child cannot manipulate or imagine a concrete situation that reflects the processes and products of mathematics" (Reeves, 1993, p. 92).

Constructivism implies that learning is deeply personal.

In Chapter 1, we see all three students making personal connections to prior mathematical understandings, and making connections to other issues in their lives as well. For example, Lorna and Rose do not share Benazhir's interest in the color-coding of algebraic expressions from the previous lesson. When Benazhir drops her idea of making separate rectangles to show a common factor of 2 in a trinomial, Rose chooses on her own to follow up the idea. Later, Lorna wonders about whether a square is a rectangle to make sense of $x^2 + 2x + 1$. Immediately afterwards, Benazhir first uses exponent notation to express $(x + 1)(x + 1)$, and sees the writing of the dimensions together as multiplication, perhaps for the first time. Although these descriptions are conjectures I have made based on the learners' words, it appears that each student has drawn on past experiences selectively and uniquely, and the result is different personal pathways to (apparently) different personal understandings. "Individuals each construct *their* individual mathematical worlds by reorganizing *their* experiences in an attempt to resolve *their* problems" (Cobb, Yackel, & Wood, 1991, p. 84, italics in original).

The strongest example of the relevance of students' particular prior experiences is not from the transcript I have shared, although it does involve Rose. Two years prior to this research, I had been observing Rose and a partner as they learned about fractions, using a "pizza covering kit" consisting of whole pieces of paper, halves, fourths, eighths, and sixteenths, thirds, sixths, and twelfths (Kieren, Davis, & Mason, 1996). After more than a week performing various tasks to which fractional notations could be ascribed, Rose was recording all she could do and all she knew about six eighths. She was in the middle of a page which had begun as a recording of all the ways to combine various pieces to make *6/8*, when apparently she saw what she was writing quite differently. She stopped, expressed amazement, said something in Chinese (she had come from Taiwan two years before), and started to write more equations at a furious rate. I asked her what she had said, and she reluctantly but politely translated. "These are *fractions*!" I shrugged that of course they were, and that we had always called them that. Rose explained further. She knew that all these activities with paper folding, fraction tapes, and the fractions kit were fractions, but it was only at this instant that she had recognized all of this to be what she had learned in Taiwan as rote numerical processes. There, she had learned, for instance, how to find the right number on top and the right number on bottom when the teacher or the text had given her two other such number combinations. For example, when she had been given *1/6 + 7/12 =* in grade five there, she could change them both into */12* numbers and then simplify to get *3/4*. And she had just now realized that the numbers which she had been writing down looked just like those previous notations. Her "delicate shift of attention" (J. Mason, 1989, p. 2), her connection to prior experience, was to recognize that what she had been doing numerically two years before was relevant to all the experiences she felt she understood so well in her new learning language. "These are *fractions*!" As she connected the words from two languages and the experiences from two school cultures into a unity, I saw her quite meaningfully as a Taiwanese-Canadian, where I had seen no reason to hyphenate her status in an Edmonton classroom before.

Interestingly, when I repeated this story to Rose in a conversation within the grade nine research program which grounds this writing, she had no recollection of the event. Apparently, in this story, Rose and I are both examples of learners making selective connections to elements of their pasts.

> What we are seeking to emphasize is the personal nature of the meaning of any new mathematical idea. A new idea is meaningful to the extent that it makes connections with the individual's present knowledge. ... It is obvious therefore that no two people will have the same sets of connections and meanings, and in particular teacher and learner will have very different meanings associated with mathematics (Bishop & Goffree, 1986, p. 315).

Another kind of uniqueness of students is apparent in Chapter 1. Although the three students are all similar and capable students in many ways and they end up each being able to do the example questions and generate similar answers, Benazhir's purposes and sense of what matters are very different from Rose and Lorna. These differences bear directly on how Benazhir deals with the learning activity, with her groupmates, and with her teacher. Rose's intellectual curiosity about Benazhir's non-standard answer to one of the questions is also a personal distinction of significance for her way of dealing with the learning opportunity. Although we can assume that such ways of learning will have experiential roots, I find it useful to view students' orientations to learning and to doing math as distinct elements of students' diversity relevant to their mathematics learning, rather than subsuming them as extensions of the diversity of experiences which they bring.

How students learn is best viewed as deeply personal. Because such a view emphasizes the relevance of each person's own unique experiences and purposes, I find it appealing to suggest that making sense is an "idiosyncratic" act, although I fear that the word carries haphazard connotations and could encourage the discounting of the aspects of sense-making that students do interactively. My use of "personal" throughout this section to describe students' learning is a compromise: although this choice may not imply strongly enough the unique nature of each learner's construction of understanding whenever their past understandings and present experiences interact, I do not want to suggest that it is only the purely unique aspects of learning that are interesting or important. In fact, the common (taken as shared) aspects of students' understandings form the basis of the students' interactions, even as the link between a student's expressed understanding and each student's sense of identity fuels their interest in each other's statements.

To teach well, when learning well involves so much of what makes each learner unique, does not mean that teachers must know everything about every student, but a teacher can best influence a student when the teacher is able to learn what that student considers relevant to his/her learning. However, traditional tests, the comparative measuring of how much each student learns, say almost nothing to teacher or student about learning processes or how a student learns well. Nor do they bring personal information about each student to view (F. Smith, 1995). Teachers must find ways to notice relevant elements, student by student, because the uniqueness of each student's learning insists on being relevant.

> Knowing children does not make teachers any more certain of how to act toward them than they otherwise would be -- whether at any moment to hang back or to push. But without eyes on the child -- and the world in the child -- one cannot make such difficult decisions wisely and might as well be teaching or evaluating by remote control (Jervis & McDonald, 1996, p. 565).

There is another pedagogic implication of a teacher's view that each student constructs understanding inclusive of that person's relevant past experiences and particular perspective. Instruction must include avenues for the teacher to show she values such personal connections as well as avenues for the students to express and develop them. Often this very personal and interpersonal element of pedagogy is left to the verbal interactions in classrooms, but when it is built into the formal elements of the pedagogy, such as activity sheets, the personal element can achieve a formal credibility, and students' interpretations of these invitations to express personal aspects of their understanding can be viewed more formally too.

Again the value of personal differences within learning activity is reflected in the questions from the algebra tiles sheets. (See Appendix B.) At first glance, the sheets appear to be dominated by straightforward practice questions. However, the questions which request responses on the back of the sheets are clearly not the standard extensions of arithmetic practice which students would find in textbooks. Generally, these questions point toward the challenge of expressing the patterns they notice, or expressing a connection from those patterns to their prior understandings without prescribing what to express or how. In this space, students were able to deal with aspects of the mathematics that intrigued them or interested them. As their answering allowed them to develop their awareness of these aspects, their answers portrayed particular elements of each learner's learning. For example, more reluctant participants tended to minimize their effort on these questions, having judged incorrectly from their format that the teacher did not value them highly or that they weren't real math. In other words, these questions successfully showed these students' orientation to do only what the teacher valued, and their view of doing math as just doing practice questions.

If we blend this deeply personal element of learning with the previously developed element of learning requiring interpretive reflection on experience, an indirect way arises to determine the relative value of an activity (or, in a particular case, the experiences it triggers). A sense of the power of the meaning-making, such as during a class interaction, could be determined by the diversity and depth of the conjectures and connections which the students propose and develop. If an activity sponsors nothing among the classroom's students but an impersonal, commonly held understanding, it is unlikely that the understanding is either deep or rich. If, however, a class discussion or a class activity such as the algebra tiles sheets derives a wide range of personal interactions with its substance, it may be considered likely that the understandings will be richer and deeper. For instance, to gauge whether the practice questions on the algebra tiles sheets were adequate and sufficient, it might be best to look indirectly at them. With the non-traditional questions which direct students to express personal meaning, it is possible to perceive the quality, diversity, and personal nature of students' answers; and, consequently, a teacher could determine by the variety of thinking which is in evidence whether the other questions that preceded these reflective questions had provided sufficient 'food for thought.'

Let me say this more directly. The task for the students must provide sufficient common activity for the students to have a foundation for discourse. Yet the activity must have sufficient complexity so that students can anticipate that others will have noticed or interpreted elements worthy of exploring in conversation. A dynamic pull between two opposing principles results, one which can structure the teacher's choices during task design, activity development, and discourse leadership. To what extent should the common ground (most likely including the intended focal concept of the task) be the target of students' attention, and to what extent should interests of particular students (most likely including elements of students' past experience and elements of students' sense of purpose as students) be fostered? Both the activity and the discourse depend on the inclusion of both common and unique interests.

Constructivism is more than problem solving.

One particular element which is sometimes associated with constructivist student activity is the concept of problem solving. In this regard, I am interpreting *problem* in its more common usage in mathematics education, as a challenge where the step(s) to a resolution are not immediately apparent (J. Mason, Burton, & Stacey, 1985; Polya, 1971;

Sternberg, 1992). I do not view the activity of Benazhir, Rose, and Lorna as presented in Chapter 1 as the solving of particular problems: the individual questions in themselves tend not to be problems, as the way to a solution, to build a rectangle and express its dimensions, is apparent. When a method is not apparent, Benazhir shows well the dynamics of specifying a particular problem and addressing her attention to it. However, Benazhir shows the limitations of viewing each question as being a *problem*: in such a frame, simply getting the answer becomes the goal. When she recognizes how to get an answer for the type of question that has temporarily presented itself as a problem for her, the inquiry ends, without addressing what the problem and its solution could represent in terms of a larger frame. Benazhir has framed her mathematical success in the school sense, where giving correct answers to assigned questions can be viewed as success.

Perhaps a case could be made that the challenge of making sense of how tiles and algebraic expressions interrelate could be called a *problem situation* (Niss, 1993b; van Dormolen, 1989), or an *ill-structured problem situation* (Silver, 1994; Wells, 1995), or a *problem setting* (Schon, 1988; Schrag, 1992) within which the students pose their own problems. This conceptualization provides room for each learner to signify the elements in question and to situate themselves within the inquiry as they choose. Although often a problem to solve offers a focus for intellectual inquiry, defining a problem and then pursuing its resolution is only a particular way to frame learning through inquiry. In a more general view, the broader challenge of understanding a topic, the intrinsic motivation of sense-making, can serve as the intention guiding inquiry-based learning without conceiving the process as beginning with students' problem posing or setting and then problem solving. Learning is motivated by the desire to understand in inquiry-oriented learning (Bruner, 1990; von Glasersfeld, 1991a).

Chapter 1 includes a strong example of the natural motivation of students to understand: when Rose wonders about Benazhir's unique way of factoring out common factors of 2, she is not motivated by school goals especially when she determined that the method wasn't my intended goal for them, and it is awkward to think that she is perusing a solution to a problem: she just wants to understand as well as she can the context within which she is active. This is not well-described when it is called *problem-solving*; perhaps it is better to call it dealing with a problematic, which implies well that our most significant problems do not lend themselves to being solved, but rather dealt with (Cobb, Wood, & Yackel, 1991; Confrey, 1991; Dewey, 1960; Dewey in Hiebert, Carpenter, & Fennema, 1996; Dewey, in Wasser & Bresler, 1996). However, I am not convinced that learning is best perceived as engagement with particular challenges, especially when by using words such as *problem* we cast those challenges as being negative elements of our environment. Instead, learning is what we do as we shape our environment and shape ourselves to it (Steffe & Kieren, 1994; Varela, Thompson, & Rosch, 1993).

I think that instruction based on problem-solving is likely to be an improvement over instruction based on rote practice, and instruction that involves students in problem-posing will foster more complex experiences and understandings than problem-solving-based instruction (Freire, 1970; Gonzales, 1994; Silver, 1994; Wilson, Fernandez, & Hadaway, 1993). Yet, each of these improvements implies more complex learning processes for the students and a relevant question is how much complexity can a teacher convince students to accept. Similarly, the provision of problematic occasions, in which even the choice of when and how to define a situation by posing a problem falls to the student(s) is a further broadening with educational benefits but pedagogic complications. "In effect, the mathematics classroom becomes a problem-solving environment in which developing an approach to thinking about mathematical issues, including the ability to pose questions for oneself, and building the confidence necessary to approach new problems are valued" (Schifter & Fosnot, 1993, p. 9).

However, it may be advantageous to design and structure learning occasions without making the primary concern its inherent likelihood to sponsor problematic events. If the occasion's activity space has complexity and richness (consider, for example, the relevance of aesthetic richness in the last algebra tile sheet, Appendix B), students will be able to interact meaningfully with the activity and with themselves within that occasion whether they actually frame their challenge to understand as the solving of problems or not. If we privilege too much the very powerful process of students defining problems for themselves and then pursuing their resolution, we may lead ourselves to underprivilege other important activity. One example is the practicing of newly learned skills, during which time students can explore the range of applicability of a skill and develop their speed and accuracy, without specifying a particular problem. Another example is the negotiation of socially acceptable statements of one's understanding, where it may sometimes be helpful in some ways and limiting in other ways to view the negotiation of meaning as a problem to be solved (understanding confirmatory statements and situating a group's efforts to establish mutually inclusive discourse procedures, for example). Occasioning of activity space must assume that the students will have significant influence on the activity and all other features of the context as they are significantly affected by and through their interactions with the context. This orientation to provide a rich activity context which is shaped and is shaped by the students may help align constructivism with biological cognitive theorists who hold cognition as an intentional and mutually determining relationship between organism and environment (Bateson, 1972, 1979; Bogdan, 1994; Bruner, 1990; Goldsmith, 1993; Johnson, 1987; Maturana, 1978; Varela, Thompson & Rosch, 1993; Wuketits, 1990).

Constructivism means attending to process as well as product.

In the set of quotations embedded earlier in this chapter (p. 53), rich, often metaphoric verbs point to learners' actions from a constructivist perspective. As well as the focus on the students in preference to the teacher, such a perspective involves a change in emphasis toward the processes of learning rather than the outcomes. In comparison to traditional reports of teaching experiments, this writing and the teaching it describes are extreme examples of this change in emphasis. Perhaps evidence of the extent of learning is available in the byproducts of students' processes, but the nature of the learning itself is available directly through interaction with and being a part of those processes. Since it is the processes that determine the outcomes, even a teacher interested in learning outcomes must be process-oriented, especially as it is only by influencing the processes that the teacher can claim to do more than measuring learning.

Prioritizing learning process over learned outcomes is easier to claim than to operationalize. Teacher-led conversations can consistently inquire about how students obtained answers, and this focus can be extended into students' written work by asking questions that direct students to look back on their inquiry processes. Within the teacher's role as evaluator it is much harder to move away from the traditional product orientation. Let me outline two processes from the assessment of the algebra tiles sheets which communicated that process was a priority through its assessment component. I will first make a claim about teacher assessment as being both information gathering for the teacher and an interactive feedback system for the learners, but not necessarily a ranking process or a mechanism for monitoring of work. Next, I will demonstrate an approach for transferring the responsibility for monitoring answers for correctness to the students, not only to allow the teacher to be more process-oriented in assessment responses but also to provide an enrichment of the students' activity by the inclusion of student self-monitoring of the mathematical activity of the classroom. I will also point to some positive side effects of this choice.

The algebra tile sheets were collected for assessment but the first priority for that assessment was not the assignment of marks. The fronts of the sheets, which generally consisted of one-right-answer questions, were not marked except for those questions which asked for the students to explain or show how a factoring process worked. Instead, I "marked" the students' work on the back of the sheets, which was where students were encouraged to do something which could reflect their personal understanding rather than simply express the expected information. "Marking" consisted generally of simply expressing in words my appreciation for anything unique or interesting, and making a suggestion or asking a leading question for the students in case they chose to develop that line of learning further. When there was evidence about the learning processes of a particular student, I reserved providing feedback until it could be confirmed directly during class time. At such a time feedback on such a complex process can be developed through conversation rather than through a one-way written comment and the student could decide how to act upon it immediately. Students did ask why the front questions weren't marked but they tended to accept my explanation that they were responsible for knowing whether they were correct or not by having constructed the rectangles. Hearing this expectation helped them to understand that answering the questions was for the sake of developing the understandings expressed through the answers on the back.

Although it was less prevalent than earlier in the intervention, students still wanted to know their mark or their score. Some felt that a mark was their due reward or pay for their effort, despite my often-expressed stance that a student's responsibility was to learn and not to work. Students valued their effort highly and wanted it acknowledged often. Whether the students were actually doing and completing the sheets was not a significant concern by the time of year that these sheets were used. completion was seldom made an at-home responsibility, and within the group structure and for personal reasons the students were motivated to engage productively with assigned activity. As a consequence, I could sufficiently monitor completion directly during class time. The students' acceptance of a different purpose for the teacher's assessment of their learning developed concurrently with their acceptance of a different purpose for their own efforts (learning rather than earning marks). Thus, marking the sheets did not have to accept the burden of monitoring the production of answers and generally students accepted my explanation that my marking had a different intention. By the time these sheets were used, most students were more interested in how I characterized their effort, as this was feedback they could think about, than in how I enumerated their effort when I marked the artifacts of their learning. Whether they knew what to do in response I will develop in a later chapter.

You will notice I do not use the term *work* in this writing to describe what students do in class or at home to learn. I did not challenge students' use of the term, and indeed would characterize most of their usage as being implicitly true to more general uses of the word. Students described their activity as work when they were engaging in producing answers as if the answers were a marketable commodity. They used the term when they were positioning themselves as if they were employees and I was the boss. It was my interactions with these students which first convinced me to avoid using the word "work" in referring to learning processes. Since then, I have found avoiding the verbal trappings of the myth that work was what school was about to be more complicated and more thought-provoking for myself than I had anticipated. I now consider the use of the word to be symptomatic of a role orientation antithetical to intentionally learning by constructing understanding.

The decision not to mark the answers of the algebra tiles arithmetic questions for correctness is not a denial of the importance of correct answers. Rather, it is a reprioritizing not only of the value of answers in relation to processes and choices but also of the roles and responsibilities for correctness. Teachers do not need students' answers to be right and, as previously suggested, right and wrong answers are of little explicative value to teachers. It is the students who need to get answers which are correct, not only for their own satisfaction but also as indicators of their progress within a learning opportunity. Thus, it makes sense that correctness of answers be a student's concern. Yet, this did not mean simply that students needed to mark their own work from answer sheets provided to them: rather, they are left with the need, the authority, and the opportunity (i.e., the responsibility) to determine for themselves whether their answers are right. When the students construct or visualize a rectangle with the area of the given polynomial, they know that they have found the factors in the dimensions. The procedure itself includes the opportunity for students to satisfy themselves that their answers generally are right, which means that they can confirm for themselves the appropriateness of their chosen or self-created procedures.

That students can satisfy themselves that their answers are correct is a huge bonus, both to the teacher and to the students, of inquiry which is situated in a natural and accessible context. In such an environment the students can turn to the context itself for confirmation of the correctness of processes and products, because a natural environment offers ways to check whether an understanding fits with the already recognized features of the environment.

> To help students gain a sense of their own voice in mathematics, we first help them move away from a strictly answer-oriented approach to solving problems. We have found that it is impossible to achieve this goal in an environment where answers are supplied; therefore, we do not tell students whether they are right or wrong. ... In explaining her solutions, the student must learn to give voice to her discoveries (Morrow & Morrow, 1993, p. 55).

Even in the simple context of rectangles made out of algebra tiles, students have the opportunity to use their prior understandings of multiplication and area to independently interpret their own processes and products as sensible. In Chapter 1, Rose and Lorna made good use of this feature to confidently and autonomously build more complex understandings of the algebraic expressions they were developing. Their confidence depended on their sense of the match between images of rectangles and the expressions which described them, without having to turn to an external authority (the teacher or an answer sheet) to provide assurance that they were correct. Benazhir displayed less of this intellectual autonomy, until she arrived at the realization that the arithmetic of algebra had the same relationships as the arithmetic she had previously learned.

It would be inappropriate to demand that students rely solely on this self-monitoring of imagery within contexts for validation of their processes. After all, students are already in a natural context, without a mathematical microworld (Tall, 1989; Thompson, 1989) such as the algebra tiles, and that context is the classroom. Just as within a mathematical context the classroom context includes natural ways of determining whether a process is appropriate. Because students' desire to know they are proceeding correctly in classroom activity is purely cognitive, the intellectual images of the microworld cannot provide all the confidence which students need. Students want to know whether their choices and processes meet with the approval of the teacher and their classmates. As you saw in Chapter 1, when a student was not satisfied with her procedure or confident with it, it was simple to check with a learning partner, or to involve a teacher. I believe that students need nonjudgmental support in their social context, for them to perceive themselves as situated well enough to take the risks associated with learning in personal and interpersonal ways (Peterson & Knapp, 1993). Confirming and amending one's choices through interaction with others, including the teacher, also allows students to confirm and amend the extent to which the context supports and encourages their explorations (Cobb, 1995b; Prawat, 1995). As mentioned earlier the challenges of expressing oneself and aligning one's understandings with others' expressed understandings enriches the meanings each student derives. Students are more able to reflect upon and consciously manage their learning processes when they can sensibly confirm for themselves that their processes are appropriate. The processes themselves become enriched and supported through the interactions by which the processes are confirmed because a reflective element becomes a natural part of the inquiry.

I have suggested that connections between images of rectangles and algebraic expressions are important for confirming students' understandings as they develop. I have also suggested that connections to prior understandings such as to numerical arithmetic can provide similar confirmation. I do not mean to imply that connections such as these are not valuable elements of the content themselves. Nor do I wish to imply that the connections are natural features of social contexts (classrooms) or mathematical contexts (microworlds such as the algebra tiles) which students will automatically develop and use. In the next section, I would like to suggest a proactive stance regarding connections with prior content for constructivist teaching.

Constructivism includes an emphasis on connections among concepts.

Although the relationships among elements of a natural context may be available for students to use to confirm the fitness of their processes on an ongoing basis, the value of confirming and the skills of doing so need to be taught if the teacher wants to rely on students to autonomously judge their answers themselves. Students need to be taught the importance of pursuing connections if not for the value of those connections for developing richer understandings then for the value of perceiving ways in which content can apply to elements of the students' lives. Connections to prior understandings provide an opportunity for students to re-view those understandings and enrich them in the light of their new experiences (Kloosterman & Gainey, 1993). In the context of algebra tiles this means that students must be encouraged and supported in pursuing connections to their primitive understanding of numerical arithmetic from their algebra experiences. Students must connect both the building of rectangles from tiles and the notation being used to record the dimensions/factors to their primitive understandings.

A difficult choice faces teachers as a result. Building connective content into students' inquiry of a new topic can significantly increase the complexity of the activity. In general, students are more comfortable learning one thing at a time in isolation from other things and if there are connections they would prefer to build those later. However, omitting connective content reduces the possibility of multiple connections developing as part of students' understanding of the primary topic, and portrays an impression of mathematics as being structured as discrete and abstract elements. If tools which support inquiry are developed within that inquiry then their purpose and their relationships to the inquiry are apparent and the impression of mathematics which is portrayed is a structure of interconnected and useful elements.

In pursuit of structures which could further connect the students' exploration of algebra tiles to their primitive understandings of arithmetic, I introduced the students to a procedure or notation we called "grid arithmetic" on the second day (the day following the small-group activity in Chapter 1). First a pair of numbers (See Figures 3-1 and 3-2) and later a pair of polynomials (See Figure 3-3) were split into component pieces additively.

	10	5	
20	200	100	
9	90	45	
			435

	10	5	
25	250	125	
4	40	20	
			435

Figure 3-1. Multiplying 29 • 15 by partial products in grids, examples 1 and 2.

	10	5	
10	100	50	
10	100	50	
9	90	45	
			435

	20	-5	
30	600	-150	
-1	-20	+5	
			435

Figure 3-2. Multiplying 29 • 15 by partial products in grids, examples 1 and 2.

Then the component pieces were added or multiplied as required with the pieces from the other number or polynomial. In each example in Figure 3-1, the two factors of the multiplication question *29 x 15 = ?* are rewritten on the left-hand and top edges of a grid. In the first example, the two factors are separated according to their digits and place value. Next, the two values are separated as they would be if shown by coins. The third case is a variation on the first and the fourth case introduces the possibility of negative addends. All four of these examples were suggested by students as we developed this notation. I value that this notation sponsors student choice and provides an opportunity to celebrate diverse thinking about how numbers make sense.

The students were able to determine how to combine the resulting components to get a composite answer. This gave them another way to re-view their understandings of multiplication. When the method is used for polynomials (Figure 3-3), the students can re-view their algebra tile answers as the actual product of two factors as well as the dimensions of a paper rectangle. In each case, a trinomial is shown as the product of two binomials, with one binomial split into two terms on the left edge of the grid and the other binomial split into two terms on the top edge. The partial products are shown within the grid and the sum is shown in the bottom right. In fact, a rectangle can be made from algebra tiles so that its visual image matches the four partial products obtained in the grid multiplication.

	x	*+ 1*	
x	*x^2*	*+ x*	
+1	*+ x*	*+1*	
			x^2 + 2x + 1

	2x	*+ 4*	
x	*2x^2*	*+ 4x*	
- 2	*- 4x*	*- 8*	
			2x^2 - 8

Figure 3-3. Two examples of multiplying binomials by partial products in grids.

The second example (Figure 3-3) includes the use of negative values which I did not build into my version of the algebra tiles kit. Yet, both the numerical case with negative terms (Figure 3-2, example 4) and this one operate within the grid in the same way as the all-positive examples, and the students had no difficulty with the possibility of negatives within polynomials despite their limited experience with it. (On the other hand, this example did activate some students' nervousness with regard to their recall of the "rules" for arithmetic with negative numbers, something that appears in another context in a later chapter!) With negative values the grid notation provides a simple framework for showing the mechanics by which difference of squares binomials can come from the product of two binomials. It also allows for extending the processes the students derive from the all-positive algebra tile rectangles for identifying the binomial factors of a given trinomial to include negative terms. However, we certainly did neither of those things on the first day. With or without negatives, this notation is useful for extending explorations done with algebra tiles to factoring more complicated rectangles, but its first purpose is more fundamental. Primarily, grid arithmetic provides a conceptual and procedural bridge to the students' prior understandings of multiplication and factoring of whole numbers. In so doing, it fosters the development of a conceptual and procedural understanding of the multiplication and factoring of polynomials.

Should this particular notation be additional content to be assigned to students when they are learning algebraic arithmetic? Should it be taught in preference to or as supplemental to vertical multiplication of binomials or a memorized system for organizing the four partial products that result when two binomials are multiplied? This is a complicated question: the grid arithmetic was not introduced as the focal point of the students' learning, but served as a lens for the students as they focused on the algebra tiles and algebraic notation for showing factors. I feel quite comfortable with my choice at the time, which was to downplay the content value of the grid arithmetic structure, and offer it as an interesting supplement to the sheets. This choice contrasts with the choice I described earlier when I first began teaching with algebra tiles and simply made the use of rectangles to show factors additional content (p. 48).

How did the students respond? The students seldom used grid arithmetic for confirmation of their algebra tile examples. The visual imagery from the tiles themselves provided sufficient confirmation for the factoring answers that they wrote down or drew. Students readily used grid multiplying when requested to do so by the sheets. (See the third sheet in Appendix B.) They were also able to use the method to conceptualize how the factors and its products were related and were able to compare the relationship this arithmetic process framed with the same relationship as framed within tiles and rectangles. Because the algebra tile context provided sufficient verification potential, the grid arithmetic system of verification was viewed by students as an additional, rather than an essential, process. This meant that assigning the clerical task of checking their answers with grid arithmetic would have been to distract them from the sense-making which they were doing. However, showing them the structure successfully provided another option for students to consider using as they pursued their understanding of algebra arithmetic.

To different extents, every group did make use of the structure at times of their choosing. On the day it was introduced, students enjoyed exploring the different ways that numerical or polynomial factors could be split into components yet still allow grid multiplying to be satisfactorily completed. Also, every group also took the time to "invent" grid addition and subtraction, and some groups tried to invent a grid division system. This latter provided little satisfaction, but the exploration of how grid addition and subtraction would work (the key was to combine or subtract each component from one side with only one component from the other, leaving other cells empty) was both spontaneous and satisfying for the learners. Students also recognized that the grid multiplying structure could visually be laid over the tile rectangles for any factoring question, if all the units tiles (bits) were put into the lower right corner of the rectangle.

I did not pursue any data on whether the grid multiplying increased the students' understanding of the fundamental structure of multiplication of partial products as the distributive relationship of multiplication over addition. Similarly it would only be speculation to believe that the adding together of partial products with the same place value (the 10^2 and the 10^1 and the 1 terms) when multiplying two numbers this way helped to clarify or add meaning conceptually to the adding together of the squares and strips and bits (the $x^2$and the x and the 1 terms) when working with the tiles (the polynomials). As the algebra unit continued, students did turn back on occasion to consider the grid arithmetic within other contexts. As Chapter 4 will show, the students often had to struggle to re-develop the concepts involved when they brought the process into later inquiry. Perhaps this is attributable to my choice not to formalize the process as content. On the other hand, because it had not been formalized as content, the students did not focus on the notation-tool itself, but came to understand it through their actual use of the tool in inquiry processes which they controlled. Perhaps in ideal contexts the primary purpose of algebraic notation of all kinds would be as a support for student inquiry into patterns and relationships.

Constructivist learning involves interaction.

Discourse is accorded a prominent place in the landscape of the constructivist classroom. Some constructivists hold that understanding is shaped only through verbal discourse by a social group (Berger & Luckmann, 1967; Bloor, 1991; Gergen, 1982). As we saw in Chapter 1, however, many elements of a learning process can take place with relative silence. Also, I do not want to ignore non-verbal forms of action, communication and reflection by according a theoretical monopoly to discourse of any particular given form. However, constructivist teachers must be concerned with students' expression of their understanding, at least because understanding can be recognized by an observer (the teacher or another learner) when it is expressed. Further, as students express their understandings, they can shape their understandings with and against the expressions of understandings of others. Social negotiation and social validation is well supported in the literature (Brissenden, 1988; Bruner, 1990; Cobb, 1995b; Moore, 1992), and is readily apparent in Benazhir's interactions with her two groupmates, and in all the students' interactions with the teacher. Discourse which includes the teacher affords the teacher a chance to guide the flow of discourse to support the teacher's own pedagogic goals for the students' development of understanding. Both the student and the teacher can attempt to limit the conversation to the mathematical content of an inquiry or expand it. The expansion can include reflective consideration of the classroom roles of teacher and students or considerations of what counts as understanding or as success in mathematics. Although I do not consider verbal discourse to be the essence of meaning-making, classroom discussion in various formats is an essential activity for meaning-making in classrooms.

Although activity can have positive outcomes which are distinct from the outcomes of the conversation during and following the activity, worthwhile activity could be assessed from its possible relationship with meaning-making conversation. Is there sufficient complexity in the task to provide a purpose for conversation, both in the ongoing shaping of the activity and in the organizing of each person's sense of the activity during reflection? Activities which have clear and non-problematic meanings do not foster rich conversations or rich understandings. Are the goals for the task perceived as reachable and worth talking about? Tasks with goals which are too abstract or too long-term will not foster purposeful inquiry-oriented talk. Does the task provided to the students sponsor sufficient commonalities to the experience that emerges to provide a foundation for conversations among the diverse participants? Even when a conversation is aiming to celebrate diversity within the social group, common ground to which each speaker and listener can connect meanings fosters listeners' interest in and understanding of each speaker's statements. Does the task provided to the classroom provide sufficient opportunities for the idiosyncrasies of students to interact with the activity, and in turn enrich the conversation?

When there is nothing but common ground, speakers have little with which to enrich the listeners' understanding, and students may not perceive a role for themselves personally in the discourse. Although rich conversation is not the purpose of a teacher's task as planned, it can serve as an indicator that the task is appropriate for fostering the construction of rich understandings.

The value of conversations in mathematical inquiry is not solely for fostering richer understanding of the mathematical activity. The theoretical literature on how understanding is constructed has yet to recognize student discourse as an experience in and of itself, beyond its role in the cognitive sense-making that follows other activity. (For exceptions, see Bateson, 1994; Gergen, 1982; Sorri & Gill, 1989.) However, we can see in Benazhir's conversation in Chapter 1 that she is not only attempting to get help making sense of the algebra. She is also clearly pursuing social goals of her own, although she does not state what those social goals might be. Perhaps Benazhir is looking to affirm her sense of how to proceed with the math sheets, or to keep from falling too far behind her colleagues, or to see herself or be seen as being much like her friends. It is dangerous to impute such motives ill-advisedly, but I put them forward to suggest that Benazhir's participation in the discourse is an experience with its own intentions, not just a part of the process of algebra sense-making which may be its topic. Because participation in discourse is situated within a social context and is likely to have personal and interpersonal intentions, it is a dangerous oversimplification to see statements about mathematics as acts with purely cognitive intentions in relation to understanding the content of the statements.

The same must be said of my interaction with the students in Chapter 1: its intentions often include the clarification of an algebraic concept or a learning process, but it will also be likely to have intentions situated in the relationships that are constantly evolving, such as fostering creative or connected thinking or establishing my approachability. Assuming any or all of these intentions without plentiful evidence may be dangerous but their plausibility suggests the oversimplification of suggesting that conversations about algebra only have algebraic intentions. Learning is always situated, and the context or *milieu* (Brousseau, 1986, in Laborde, 1989, p. 33) which is simultaneously a classroom (a locale with institutionally prescribed roles) and a social world, not just a cognitive one. When ascribing intentions to actions within such a context, isolating a single intention is risky. I will return to this theme in later chapters.

It is a natural pedagogic application of this line of thinking for teachers to assume the validity of students talking during activity in class. With so many intentions interacting with each other, it makes sense to believe that student talk which is neither monitored nor directed by the teacher can as often enrich the experiences of those engaged in mathematical activity as dilute those experiences. Even as a distraction, off-task talk may be a valuable barometer for the teacher to judge the appropriateness or richness of the activity she has provided, in relation to the students' current priorities. From such a perspective, a valid response to misdirected conversation that does not return to the teacher's intended activity space might be to attempt to enrich the activity (or any of the component elements which a teacher can influence), rather than attempting to extinguish the evidence that an activity isn't mentally engaging the students. If, indeed, the fault is with a student's motivation or lack of intention to learn, that also is worth recognizing so that the teacher can begin to construct activity within which the student can address that element (learn to want to learn). This latter claim suggests that motivation to learn is learnable, and by that I mean that it is subject to constructivist interpretation and teaching, too. This claim will be elaborated in Chapters 6 and 7.

Constructivism can guide teacher decision-making.

Learning is too complex to reduce to a single essence and is too dependent on the involvement of the learners to be viewed as a direct outcome of teaching decisions. A plausible teacher reaction to a suggestion that learning is necessarily active, grounded in each learner's past experiences, personal to the point of being idiosyncratic, and situated in and shaped by interpersonal interactions would be to decide that learning as a process is relatively unknowable or at least not amenable to being shaped by a teacher's choices. Such a view would make teaching effectively very difficult, or at least would require that teaching be defined in terms other than learning (control and management are two possibilities). It is impossible for a teacher to understand or direct learning by basing actions on the bland truths drawn from generalizations about most learners and most contexts. Thus one of the corollaries of constructivist epistemology is that teachers must get to know how their students each learn. A central feature of good constructivist teaching is that it sponsors a flow of data about the students, their values and their orientations and their behaviors and their learning so that the teacher can know them well enough to negotiate effective engagement with the learning opportunities she provides. Although generalizations are insufficient, learning is quite knowable just as people are knowable.

When the personal and complex nature of learning is accepted, a teacher's inquiry into his students' learning can be rewarding and can inform pedagogic decision making.

> The difficulties inherent in an inquiry-based environment are numerous. The demands on the teacher to be a lifelong learner, to serve as a resource, to share authority for knowledge, to set the curriculum agenda aside when necessary, and to question and learn with the students necessitate a major shift in focus on what constitutes the teacher's role. The new role suggests that the relationship between the teacher and students be one of collaboration and dialogue, where both teachers and students work toward their own growth in understanding (d'Ambrosio, 1995, p. 772).

Given that learning is too complex and too personal to manage directly, the liberation to take pedagogic action comes from acknowledging that teacher decisions must necessarily be based on incomplete information. Such action must be based on good intentions, including the desire to have more information about the complex thing which is student learning. Inasmuch as each student's learning is a personal construction not all of which is evident to any observer, especially a busy teacher, teachers must accept that they cannot custom-design curriculum for each student. Rather, teachers can learn from their students enough about how they each learn to be able to provide a broad activity occasion and within that occasion negotiate personal engagement by each student. Also, they must rely on the active involvement of the students in shaping the tasks they are offered to their own needs. In this regard, I find it less idealistic to believe that each student can be encouraged to construct for themselves a way to successfully engage with an activity than to believe that a teacher could customize individual activities to the personal backgrounds and cognitive needs of each child. However, my sense of optimism with regard to such a possibility is dependent on fundamental commitments by both the teacher and the students:

-->The teacher must provide an occasion within which activity structures have sufficient richness for diversity, and must engage with each student in support of his/her personal pursuit of meaningful experience.

-->The students must understand their opportunity and role as being to personally engage in the building of meaningful experience. Such understanding must be viewed by the teacher as a desired curriculum outcome, concurrent with mathematical curriculum outcomes. The construction of such an understanding can be considered to be subject to and informed by constructivist theory, just as mathematical understandings are.

In the chapters to follow, I will be attempting to clarify possibilities for more successful learning of algebra by designing curriculum and implementing its instruction within the constructivist frame which I have described here. I will also be suggesting that it is a natural consequence of the pedagogic orientation I have described to declare as curriculum space the students' epistemological understanding (sense of how they learn and what counts as the desired outcome of learning) and the students' ontological understanding (sense of their identity as learners and students and their sense of their roles in relation to others). That will mean that the constructivist frame which this chapter describes can clarify possibilities for implementing instruction which assists students in furthering these understandings. Finally, I will be connecting this student progress with the students' sense of their progress within their institutional relationship with the school.

In the next chapter, I will describe one application of the claims described above to the challenges of designing curriculum for students of algebra. The conceptualization of a mathematical activity will be described and the learning of one small group will be detailed as a way to explore further the claims made in this chapter.

Chapter 4. Rectangles Families

This chapter describes chronologically a three-period strand of instruction and student activity which provided the title of this chapter. After discussing the development of the rectangles families activity space as curriculum, the chapter chronicles the introduction of the activity to the students. When the activity switches to small-group inquiry, the focus switches to the inquiry processes of one group of four students. To develop the curriculum development, instruction, and student activity as a particular application of the constructivist principles described in the previous chapter, particular teaching issues are developed within the context of the description as they arise.

Some readers may choose to focus on the algebra within the activity and may perceive the complexity of algebra generally or the concept of variable in particular in a more complex way. An appendix including manipulatives for readers' use is included. (See Appendix D). Teachers of algebra may read the chapter with the goal of trying out the algebra context and activity. The chapter portrays the activity as well designed and effective for the group of students whose learning is the final focus of the chapter. The chapter which follows this one will stay with the same instructional activity but will investigate how other groups handled the learning opportunity. It will provide a broader perspective regarding the complexity of achieving success with students with a wide range of orientations toward their role in inquiry based learning, and the importance of helping students learn roles which match the teacher's expectations for their involvement. Framing this latter theme will become progressively more central as the chapter proceeds.

Variable as a mathematical concept

> Most mathematicians believe that what gives mathematics its power is the manipulation of symbols standing for anything and having no context. In a problem, *x* is *x* and nothing more. However, I believe mathematics could be even more powerful by retaining some recognition of what the symbols stand for and gearing the approaches used to that. If, for example, you are dealing with *x*'s referring to numbers of human beings, you should only be seeking integer solutions and selecting solution methods accordingly (Ascher, in Ascher & D'Ambrosio, 1994. p. 39).

> All of Diophantus' algebra consists in looking for quite determinate numbers. ...

> From the Greek standpoint the symbol *s* can only refer to a specific unknown number. Variables by contrast do not stand for specific unknown numbers. As their name suggests they stand for a whole varying range of values which obey some rule of law (Bloor, 1991, p. 114).

> The meaning assumed by the two syntactically identical formulas [for the circle, $x^s + y^s = r^s$ and $a^s + b^s = c^s$] essentially depends on whether the letters used in these equations are interpreted as *variables* or as *constants* (an interesting example of *ambiguity* within mathematical symbolism) (Borasi, 1992, p. 23).

> The essence of modern science, as it is done by scientists, is to understand the relationship between variables (Goldberg & Wagreich, 1990, p. 65).

> Basic algebraic skill must be conceived as encompassing more than symbol manipulation. Of fundamental significance are an understanding of concepts such as variable and function (House, 1988, p. 4).

> *Purposes for algebra* are determined by, or are related to, *different conceptions of algebra*, which correlate with the different relative importance given to various *uses of variables* (Usiskin, 1988, p. 11).

> Whether we look at variables from the viewpoint of the roles they play in algebra of the ways that students operate with them, it is clear that the 'concept of variable' is a multi-faceted one (Wagner & Parker, 1993, p. 123).

To be able to do arithmetic with variables could be considered mastery of the unknown. Despite not knowing a specific value, even without putting limits on a range of values, one can manipulate that value or values. Thus algebraic arithmetic becomes the foundation for equation-based problem solving, ranging from the hypothetical age, coin, and sum-of-consecutive-odd-numbers story questions of our schooling to the burn-time of a satellite's booster rocket. As well, when variables have referent meanings in a common context that we anticipate encountering frequently, the relationships among those meanings can be shown and manipulated. Thus algebraic arithmetic is the foundation for the use of formulas generally. To understand algebra, then, means to perceive variables and the expressions which include them as meaningful referents to unknown numbers and sets of numbers, as meaningful referents to objects and actions, and as subject to manipulation by arithmetic.

The algebra tiles strand of instruction described in Chapter 1 provides a narrow context in which rules for multiplication-based arithmetic with a variable can be seen to make sense and can be learned. Within the fundamental idea of ***length times width makes area***, students can use algebra tiles to bring concepts involving the multiplication of two factors into view and into hand. However, how well does that context develop the breadth of algebraic understanding which this chapter's opening quotes express? Consider the meaning of the variable in the context.

Algebra tile kits are simple for a teacher to design. First, a length to represent one is chosen. I have found 1 cm inconveniently too small for students to look after and 1 inch too large to allow a kit to be cut from a single sheet, so I select 1.5 or 2 cm. Next, a length to represent x is chosen. It should be a value which does not suggest that x is a particular value so I avoid multiples and half multiples of the unit value. Usually, to try to preserve the idea that x can vary, I provide different students with different kits, so that those with a kit photocopied onto blue paper have an x of perhaps 4.7, those with green have 5.3, and those with yellow have 5.8. Regardless, the value of x for each kit must be set by the designer, who is the only person for whom x varies conceptually. For the students the meaning of x that they receive in their hands is a single length. While they are learning the arithmetic of binomials with the concrete representation I am providing, they are dealing with x not as a variable, but as an unspecified constant value.

That algebra tiles provides for students only a single value for its variable is not a condemnation of algebra tiles. No one context, no one concrete representation, could hope to capture all the elements of all the concepts of algebra. "Variables have many possible definitions, referents, and symbols. Trying to fit the idea of variable into a single conception oversimplifies the idea and in turn distorts the purposes of algebra" (Usiskin, 1988, p. 10). As we search for active and interactive instructional procedures to replace traditional instruction, we need to acknowledge that activity within a single concrete representation is still too linear an approach to complex concepts. A rich understanding relies upon a diverse experience base, and that relies on activities in multiple contexts (Schifter, 1996; Simon, 1995; Wells, 1995).

This is an example of a larger claim developed in part in Chapter 3. Constructivism suggests the value of activity and experience beyond simply hearing how to do something and then practicing, and so the use of algebra tiles is one step forward in the development of mathematics education. Just as within the delivery metaphor for instruction it is a false belief that there is one example which if explained clearly will give students an understanding, it would be a false belief within a constructivist metaphor for learning that we could find or design one concrete representation of a concept to frame students' activity. Perceiving our pedagogic goals to be the search for the one concrete representation or activity that captures the essence of a concept and portrays it for the students is simply a transference of our former beliefs in clarity and precision within the delivery metaphor, where just the right explanation at the right time delivered in the right way was believed to convey the essence of an idea to a learner. Instead, constructivism is better grounded in an epistemology where a person's understanding of any content is based on complex connectedness among elements and to elements of other content, with both the elements and the connectedness among them emerging through multiple and varied experiences of the learner. Our pedagogy must match not only the psychology which constructivism suggests but also its epistemology: if understanding is the meaningful interpretation of personal experience, then generalized understandings will depend on the meaningful interpretation of multiple experiences (Driver & Scott, 1995; Noddings, 1994; Steffe & Kieren, 1994; Thompson, 1988).

Yet how can teachers provide for a breadth of experience when it comes to algebra, and, more specifically, with the concept of variable? Multiple experiential approaches are clearly available for mathematical concepts that deal with specific amounts: with fractions, for instance, there are circular, square, and linear fraction kits, and references to life experiences with hours of time, money, and pizza (Kieren, Davis, & Mason, 1996). However, it seems almost a conflict of kind to imagine concrete experience with an abstract or varying or unknown amount. Could that be at the root of claims that algebra is difficult to teach?

It is not as difficult to find realistic experiences in which algebraic expressions can play a significant role. As Chapter 2 describes, the first strand of instruction consisted of specific contexts in which regular patterns could be noticed and expressed algebraically. Although students made sense of and expressed their sense of the patterns through everyday actions and language, they also came to use the forms of expression particular to algebra. Students found charts, graphs, and arithmetic expressions useful in their inquiry about the patterns and in their descriptions of what they understood. Yet although they often had concrete models which showed varying specific values one at a time, throughout those processes they did not have a concrete model which represented an unspecified numerical value.

In each case the students brought variables into use in somewhat the same way. One variable name related the values of the independent variable to the context--for instance, *numpeople* or *p* might represent the number of people in a room. Another variable name was coined to refer to each of the values derived within the context by considering a single particular value for the independent variable--*numhandshakes* or *h* might represent the number of handshakes among all the people in the room. Students could derive arithmetic expressions showing the relationships between independent and dependent variables in such a context, and they could recognize that the variable was a referent for particular numbers and for a particular idea simultaneously. Yet they needed more breadth in kind of experience to construct an understanding of the complexity of the concept of variable. Also, they had not yet encountered any context which called for manipulating algebraic experiences arithmetically.

The pedagogic challenge was to design an inquiry activity which would expand the kinds of experiences which students had regarding the nature of variables. The patterns strand provided students with an opportunity to see the use of variables as sensible, but their use of variables was as a tool for viewing the details of the context rather than considering variables themselves as the object of their analysis and contemplation (Sfard & Linchevski, 1994). I wanted to provide in their next experience an opportunity to make variables and their use the object of their attention, rather than the lens with which they attended to the details of a context. This distinction is similar to having students with semantic understandings of written language begin to notice the syntactic aspects of the language that they have been using (Booth, 1988).

Teaching students about variable

> The usual prealgebra curriculum in grades 7 and 8 pays no attention to introducing and establishing the concepts of variable and function (Demana, 1990, p. 125).

> Much of the power of mathematics comes from its extensive use of symbolic notation -- not only to identify objects, but to describe operations, and moreover, not only concisely but *suggestively*, for it is in the way in which symbolism can lead us to new results that its power lies. Yet this great use of symbolism is double

> edged: it gives mathematics its power, but it also makes the subject difficult to learn and to teach (Howson & Mellin-Olsen, 1986, p. 11).
>
> The concept of variable is more sophisticated than we often recognize and frequently turns out to be the concept that blocks students' success in algebra. Students need to become comfortable with variables in numerical contexts before they begin a formal study of algebra (Leitzel, 1989, p. 29).
>
> There is a huge difference between expressing your own generality and doing someone else's algebra. ... Young students in their early exposure to algebra can often express a general rule but find it difficult to see that expression as a manipulable object (J. Mason, 1989, pp. 3-4).
>
> We are convinced that the most important reason for not understanding mathematics is bad preparation on the concept of variable. By slowly and gradually letting the pupils get acquainted with this phenomenon of variability, closely connected to phenomena that they know in reality, we put the idea in their heads, so to speak (van Dormolon, 1989, p. 96).

The activity strand which attempted to supplement the other strands in terms of the role of variable and the contextual role of algebraic expressions came to be called "Rectangles Families." Its general intention was to have students derive relationships among grid-paper rectangles of approximately the same sizes, and to express those relationships with numerical and algebraic expressions when describing specific instances and general cases for each relationship. Basically, each group of students received or made a rectangle of a particular character, and then identified certain aspects of the rectangle which they could count as well as calculate or derive from other aspects. While numerical values and expressions could describe a particular rectangle and the actions and ideas of the students with that rectangle, to talk about other groups' rectangles as well (in other words, to make generalized statements) variables and variable expressions were needed. Describing the common-sense observed and derived relationships which the students found common in all the groups' rectangles families would require the interactive negotiation of variable names and the acceptance of a common syntax. Thus from a physical representation of particular numerical values and relationships, general values (expressed with variables) and relationships emerge through discourse.

This activity as planned reflects the constructivist principles outlined in Chapter 3. It had an inquiry base which provided students with the opportunity to construct purposeful experiences. Students were encouraged to form images for their actions on the physical objects provided and to express those images interactively in a variety of forms and a variety of social contexts. When those forms included the algebraic forms which they had used and would use in other contexts, the students had opportunities to develop and express personal connections to their experiences in those contexts. Ideally the structure of algebraic forms and variables could become the object of the students' attention as they were guided to reflect upon the relationships among those expressions.

This strand lasted for three instructional periods. The first two periods were primarily teacher-directed whole-class conversations about the rectangles punctuated by small bursts of teacher-directed small-group activity with particular rectangles. The final period was primarily small-group inquiry. The next section describes in some detail the introduction and development of the rectangles families activity, to provide a context for interpreting the learning activity of particular groups of students, both in this chapter and the chapter which follows. It also describes an attempt to lead students to construct particular understandings according to the principles of constructivist learning.

The Parent Rectangle and negotiating variable names

To start, each group was given grid paper (sheets marked in one-centimetre squares) and scissors. Each group was told a specific square to cut out, to be designated the Parent Rectangle. For instance, one group was told to cut a 13 by 13 square, while another group cut a 16 by 16 square. In the first stage of the activity, students only used their Parent Rectangle, quickly determining some ideas that could be counted and/or calculated, and in a class conversation compared their ideas with other groups.

Readers may turn to Appendix D to find a rectangles family for their use. For now, only the square of a unique color is needed. The Parent Rectangle in the appendix measures 17 by 17, but just as for each learning group within the classroom where the research took place, the readers' Parent Rectangle is unique in its size, although all groups had squares with integral dimensions ranging from 10 to 20.

I also had a square made of translucent red plastic, which I placed on the overhead projector. Its red shadow was projected onto the screen, and I told them I knew the dimensions of the square itself, but not of its shadow. Although I kept my square as a secondary element in the conversation, for me it represented two meanings of a variable: on the screen, the shadow's dimensions could be a wide range of unmeasured values (depending on the position and focus of the projector). On the projector, the dimensions of the plastic square represented that classic meaning of the variable, an amount the teacher knows but won't tell. Since there were no grid squares drawn on my square, its precise length remained unknown to the students whether they could see the plastic square or not.

In the following transcript chunk, the students learn that each group has different squares to call their Parent Rectangle, and learn to talk about the ideas, beginning with the number of grid-squares in their Parent Rectangle. The initial interplay makes apparent not only the range of numerical values but a range of words and forms to identify the ideas of the Parent Rectangle's dimensions and area. The transcript begins with me attempting to establish the "___ by ___" syntax for describing the dimensions of the Parent Rectangles, while the groups notice that the others' Parent Rectangles are all squares, but different sizes. Benazhir, Rose, and Lorna have a 19 by 19 square, but generally individual students (designated by other two-letter abbreviations, while "S" signifies unidentified students) are not so much the focus of this chapter's transcripts from whole-class lessons. Comments which are indented from the left were selected for inclusion despite being audible on only one or two of the seven tape recorders in the room, and can be considered student-to-student asides.

MrM: [Mr. Mason] I've got a square and you guys have each got a square, Your square has little squares on it, See you can tell me how big your square is.
S: [a student] 14.
MrM: 14, or 14 by 14?
S: 14 by 14.
MrM: Anybody else got 14 by 14? How about you guys?
S: 12 by 12.
MrM: 12 by 12?
S: 13.
MrM: You've got 13 by 13?
Rs: [Rose] 19.
MrM: 19 by 19. So everybody's got different squares. My square [the shadow on the overhead projector screen] doesn't have any lines on it, and if I used your squares, [holding a transparency of grid squares up to the shadow] I

could even count them, right. Unless somebody bumped the table, and it would get bigger, or they'd bump the table, and it got

S: Blurry!

MrM: Blurry, and smaller. So my square doesn't really have a precise size, it keeps changing. I could measure my square down here [the paper square on the projector]. My square is smaller than yours. Matter of fact I even cheated, I made sure my square was a particular number of little squares. I don't want to tell you because it wouldn't be an unknown number any more. People have all got squares. What can you tell me about your square?

S: It's 13 by 13.

MrM: It's 13 by 13? When you tell me 13 by 13, you're telling me two things. What is the first 13 about?

Rb: [Ruby] How long it is.

MrM: How long it is? And what's the second 13 about?

Ro: [Rob] How high it is.

MrM: How high it is. Benazhir can you tell me your, your two facts for your?

Bz: [Benazhir] 19 by 19.

MrM: Yeah so you've got the same facts right. . I'm not looking at you to tell you to be quiet, I'm looking at you so I'll know when it's my turn again. There's lots of ways and words we can use for that, right. Do you use what words, for?

S: Long and high.

MrM: Long and high? What other words can we use for that?

Bz: Length and width.

MrM: Length and width.

Mo: [Monuel] Altitude and base.

S: Base and height.

MrM: Altitude and base. Base and height. It seems strange though to have two words for the same number. How many different numbers did you tell me, Benazhir?

Bz: I told you the same one twice.

MrM: Yeah, that's right. The same number as you guys? Did you guys have the same number? Your length and width are different? You have the same number. There's really only one idea on there. Why don't we just pick a word and say we're going to use this word to tell how long one of these edges are.

S: Base and height.

MrM: Base and height's two words, unless you spell it all together, *basenheight* [prints on board].

S: Axis?

MrM: Axis. Great idea, because on a graph we have an axis here and an axis there. But well I'd save it, I think I'll avoid the word axis.

Hl : [Hilde] Side.

MrM: Side. Let's see, if we use the word side, what if I tell you to go to the side of the room. Ah heck, everybody will know what I mean. Hilde gets to choose.

This initial conversation about all the vocabulary which could be attached to the length of the side of a square is quite important. On one level, it is a matter of engaging with the various students' prior knowledge about the ideas and the words they are comfortable using. Negotiating terminology and ongoing issues of terminology help make apparent students' understandings of the underlying concepts for the teacher's consideration, and it provides opportunities for the students to open connections to prior knowledge. It is also a matter of encouraging the students to keep the idea in mind, rather than the particular term: whereas a person who understands the roles of variables in algebra will recognize the term as standing for the idea, it is not hard for a student to try to memorize a term and its related formulas instrumentally because they consider the term itself to be the key content to be learned. This conversation discourages that orientation. However, of greater importance to me is the question of who controls the naming of the ideas that arise. If the students control the terminology, this lesson can more clearly centre on the students' understandings, rather than on mathematics as a pre-defined symbol system (J. Mason, 1989; Siegel, 1995; H. A. Smith, 1995). The students' negotiation of variable names symbolizes the prioritizing of students' construction of understanding and places language in its role as a tool for developing and expressing understanding rather than as the first focus of a student's learning.

Teaching which incorporates negotiation of terminology (or variable names in this particular case) does have disadvantages. For instance, dealing with students' vocabulary from various relevant experiences and deciding which terms to use takes considerable time. However, as the end of the above transcript shows, to limit the extent of discussion, a particular student can be chosen as arbitrator. Furthermore, it is in that kind of discussion that students can come to understand the role of the terminology and the conventions or rules which might govern its use. On a different note, negotiating terminology with students makes it hard for a teacher to make up student sheets in advance or to use commercial prepared sheets or texts, since it is impossible to be sure what terminology will emerge from the classroom discourse. A sheet which is not committed to particular terminology must be more open in its structure, but is less likely to be mis-interpreted by students to be a straightforward set of practice questions to teach a skill to go with specific teacher-selected terms. Appendix E shows the record sheet provided for the students on the third day to guide each group's inquiry and expression of the relationships among rectangles families. Readers may judge for themselves whether such a sheet's effectiveness is significantly affected by the commitment not to use particular formal vocabulary of algebra before the students have reconciled their understandings of what the terms might identify.

A third concern is that student-negotiated terminology also makes the conversation more complex for the students: instead of having a single label presented to them for them to connect to a single idea, they are dealing with multiple ideas and multiple labels until the selection of a label is resolved. However, as suggested previously, understanding an idea must involve more than hearing and remembering its details. The negotiating of a variable name in the face of multiple meanings can provide students with the opportunity to recognize that the concept has different shades of meaning in different contexts, including contexts from their past experiences.

Might such complexity be too much for some students? In this case the negotiating was about the simplest of ideas from the context, the length of a side of a square: any complexity was only in the process of negotiating the label, not in the cognition of the concept. Repeatedly I have found that students doing an inquiry like the flexibility provided by a full-word variable name as compared to a single letter, because it is easier to keep the meaning of the value in view when using it. This is especially true when there are multiple ideas being developed and multiple variables must be kept sorted out. Yet student frustration can arise due to the complexity or ambiguity of facing student-negotiated vocabulary along with the messiness of an activity's context. When it does, the opportunity arises to help students consider whether math (and learning math) is inherently sequential and well-defined, or complex and personal constructed, requiring the participation of its user (or learner) through personal and interpersonal negotiation.

This completes a relatively theoretical defense of negotiating of vocabulary generally and variable names in particular as they arise in discourse. Empirical support for the claim will appear later in the chapter in the form of transcripts from a group of learners whose remarkably successful inquiry includes deciding on variable names for themselves. However, the chapter which follows will feature the voices of two other groups during the same inquiry, and the disadvantages which the above paragraphs mentioned do arise. The students to different degrees are unable to recognize the significant reasons for all the ambiguity in the task. Their belief that teachers should present mathematics to students clearly and without ambiguity is not compatible with open student negotiation of variables. This is not the only issue to arise during the rectangles families, however. In the following section, the interpretive description of the introduction of the activity continues.

The Parent Rectangle, relationships, and formulas

In the following passage, after Hilde has selected our variable name, I introduce a technical complication to the process of variable-name selection. I declare that we will add colons to the end of words which are being used as variables, in a manner similar to LOGO programming language (Kieren, 1984; Olson, Kieren, & Ludwig, 1987; Papert 1980). This represents an attempt to point toward the careful distinction between a variable as a referent for a concept or relationship (just as any word) and as a placeholder for numerical values (Booth, 1988; Foyster, 1990). Conceptual meanings can be kept in view when whole-word variable names are used, but students may then have difficulty recognizing the formal qualities of variables when they have a less alien appearance than the classic *x* or *y*. When it is flagged either with all capitals or with a colon students can recognize the formality of a variable, even if it is a word which carries semantic meaning as well. Yet at the time I explain little of this to the students: their use of variables is so limited that an explanation of the practicality of the notation would be premature. It is significant that the use of teacher authority to manage formal elements caused no obvious difficulty, but the sharing of that authority by making students discuss and select variable names caused some degree of role dissonance for students.

Also in the following portion of transcript, I engage the students with a simple relationship definable in terms of the variable under discussion. The numerical relationship between the length of the edge and the number of individual squares inside it is a relatively simple idea. Students can determine that relationship for their particular Parent Rectangle and can be helped to see the power of a general expression to describe the relationship for all groups' rectangles. In my directions for students the phrase "the number of squares" turns out to be less unambiguous than I had anticipated, but the relatively open style of discourse allows the students to check if their understanding is truly the common understanding, and the difficulty is cleared up. I also found it necessary to help the students in this class see that area was available not only by using a formula, but by counting the centimetre squares.

> MrM: So we're going to use the word *side* for this idea. Now can I get you to do one thing for me? Whenever we're going to talk about an idea, we'll just spell it that way [writes *side* on board], but whenever we're going to talk about a number that goes with an idea, we'll just spell it that way [adds colon to the end.] So I ask you, now when I say it it sounds the same, when I ask you for the value of your side, you're going to say, 13?
>
> S: For ours.
>
> MrM: Yeah. But if I wanted to tell you on paper I want the value of your side, I'd put a colon on it, to say, I want the number. Okay? What other facts can you tell me about these squares? . Now I can't go, *side: equals 19*, can I? If I did that, I'd only be making three people in the room happy, and everybody else would disagree. But we can talk about the idea. What other ideas can we talk about with a square?
>
> Kl: [Kelsey] All the squares?
>
> MrM: Lots of little squares? Okay. I'm going to talk about the squares, the number of squares, see that colon means I want a number? [writes *squares:* on the board] Did you multiply them out while we were thinking, or did you say, I won't bother.
>
> Hl: [Hilde] You mean the little squares? Or all squares?
>
> MrM: The little squares, yeah. All the little squares.
>
> Hl: All the little squares, like, the little cubical ones, right?
>
> MrM: Yeah, if you counted the little 2 by 2 squares, the 3 by 3 squares, you know what, you'd have another math question that makes a great zzupp [this is a word we had coined earlier to describe the flow of functions from left to right on Cartesian graphs] when you're done. I won't mention that though, because you guys would say oh oh, there's more of those Zzupper-stuff. What's your value for squares?
>
> S: 14 by 14. I don't know.
>
> MrM: Well just count them.

S: 196.

MrM: 196 for you guys. That's right for you guys. [various answers audible.] 169 and 144. Okay. Um, you guys know *12 times 12* off by heart don't you? Do you know *14 times 14* off by heart?

Ca: [Carlos] It's up there.

MrM: Oh! Carlos says, I don't know it off by heart, I know it off by back, and he turns around. These guys gave 19 by 19, Carlos can you give them a hand? [Carlos refers to a chart on the back wall of the classroom which lists the squares for the first forty whole numbers.]

Ca: 361.

MrM: 361. That saves them some counting, right. Can somebody tell me the way to calculate the number of squares if somebody tells you the number of a side?

Hl: Just for ours?

MrM: Okay, use your number, and tell me how it works.

Hl: *16 squared.*

MrM: Okay, *16 squared.*

Hl: Or *side: squared.*

MrM: Or *side: squared.* That way we can talk about all the squares in here, even mine. I haven't measured my side yet, or I haven't told you guys my side yet, but if somebody tells me like this, I can just go *side: squared.* Is that going to work for all squares, if I cut a new square out?

Hl: Yes.

Sv: Yes.

MrM: Does it work for all squares? Okay, so we call it a formula, and it works, right?

Clearly, the number of grid-squares within the square Parent Rectangle is easily counted and easily calculated from the given dimension. Here, a calculation process can be confirmed by counting to provide correct values for a given instance, and a formula can be recognized as summarizing that calculation process (i.e., the relationship between two concepts, in this case the length of a single side and the total area) for any instance with specific common features (i.e. for any square).

Of course, to express a calculation process as a formula depends on the use of terms for each of those ideas. We had *side:* for the length of the edge, although later this became *sideorig:*, *sideOS:* (pronounced "side-O-S"), *edgeOS:* and *sos:* in quick succession (all shorthand for "side of the Original Square," as the Parent Rectangle came to called), when the students saw they had to distinguish between the dimension for the Original Square (Parent Rectangle) and the other rectangles they made later. On that day, we had many suggestions for the other idea: *minisquares:* , *numsquares:*, *numsquaresOS:*, *minisquaresOS:* and *MSOS:* all had their adherents, although both *areaOS:* and *squaresOS:* had the most consistent following.

In the transcript below, some students find themselves confused when faced with the relation, *anttripOS: = 4(sos:)*, describing how far an ant would travel making a trip around the outside of the Original Square. These are the students whose struggles the next day with small-group inquiry with rectangles families will play a significant role in the next chapter, so it will be worthwhile to hear from them in the context of this teacher-centred lesson. Kelsey could be counted on to speak out when she or her group-mates, Colleen and Ruby, were confused. As the asides in the transcript below suggest, some students are frustrated this time by Kelsey's decision to interrupt the flow, while for others Kelsey is expressing a confusion they themselves feel, and others are simply entertained by the diversion. For me, a student's right to say, "I'm confused," is sacred, and my support for Kelsey is relatively independent of the pedagogic value of her query to her classmates. We pick up the discourse just after we have finished discussing that *areaOS:* $= (sos:)^2$ and we are looking back at *anttripOS: = 4 (sos:)*, which was written on the board at an earlier stage but not explained. Is Kelsey's confusion a request for a variable that is easier to associate with its meaning (as I interpret it), a request to be reminded of its meaning (as Colleen interprets it), confusion about what the formula means, or an attempt to match this relationship with Kelsey's prior understandings of how to get the distance around something?

MrM: Okay?

Hl: Four side original square? Oh! Now it makes sense.

Ma: What's *sos:*?

MrM: Side original square.

Kl: [Kelsey] What is this 4? Side original square?

Cl: [Colleen] That's the ant-trip, Kelsey. It walked around the edges of the square.

MrM: Now we've got somebody fighting over the other label here.

Rb: [Ruby] No, that one's fine. You call it what you like it.

Kl: *a t o s:.*

MrM: *anttripOS:*? Guys, *anttripOS:* Guys, *anttripOS:,* we've got a complaint.

Hl: Another one? From who?

MrM: Guess. [laughter]

Rb: [laughs at something Kelsey whispers to her.] She wants to call it *kelseyOS:*! [more laughter]

Kl: No I don't.

MrM: What do you want to call it?

Kl: *perimeter:.*

To: [Tony] That would be a good word.

Ma: [Martin] Oh, now I see.

Rb: Oh, just leave it, Kelsey.

MrM: Well if it doesn't have OS on the end, we'll get really messed.

Hl: That's boring. I like *anttrip:* Kelsey.

Kl: I didn't complain! I didn't complain. I just xx.

To: How about *POS:*?

MrM: A good mathematician should complain a lot, so that's why I'm supporting her on this, I'm not going to let her take it back.

Later transcripts suggest that the source of Kelsey's concern may have been that her primitive understanding of perimeter was to add up all the sides and multiplying was for area. She did not perceive that multiplying by four was essentially equivalent to adding the side up four times. Thus, she was unsure whether to connect her understanding of perimeter to the ant-trip concept as it had been developed. When Kelsey suggested *kelseyOS:* I do not think many students recollected that it had been she who had first suggested the distance around the outside as an idea to be investigated: they laughed and I went with the flow, succeeding in ensuring no one was pinned down. We didn't deal with the detail of agreeing on a single formal label that day. Yet the conversation is rich with conceptual value: we had been engaged in conversations in which variables represented both a conceptual element from the context of the rectangles being used, and a convenient label for a replacement set of numerical values. Considering possible variable names meant thinking about what uses were anticipated for that variable. Also, Kelsey was pursuing connections between today's ideas and her understandings of related ideas. Variables representing values within relationships were becoming part of the students' personal toolkits.

In practice, the students did not tend to need variable names to discuss an idea generally. When students expressed their answers to a question about a concept in numerical terms specific to their own Original Square, other students in other groups tended to have no difficulty translating those numerical expressions to the numbers relevant to their particular Original Square. For instance, for the distance an ant would travel in a trip around the rectangle, when a student in one group said "$14 + 14 + 14 + 14$", students in groups with different squares heard the idea being expressed, to simply add the dimensions. A member of the group whose Original Square (Parent Rectangle) was 12 by 12 added to the previous answer by saying, "Or you could go 14×4", quickly translating the numerical values they had used to the case being discussed. The next comment, "Technically, it's $2 \times 14 + 2 \times 14$, because it's supposed to be a rectangle," came from the group which had a 16 by 16 Original Square. Students were specifying numerical values, but their meaning, both intended and received, appeared to be general.

The final student statement in the above paragraph sponsored a quick discussion about how students remembered their perimeter formulas, and whether $2L + 2W$ was the same as $2a + 2b$, and whether one was allowed to say $2L + 2W$ if $L$and W were identical or if one had to switch to$4s$. I suspect that I closed this aspect of the conversation rather quickly before the issue was resolved for all the participants -- in retrospect, I view this discussion as being a meaningful parallel to the excerpt which featured Kelsey's questioning of $anttripOS: = 4(sos:)$.. Perhaps I sensed that closure on the issue would be premature for some students at that point, but it is more likely that I simply wanted to get on with working with the rectangles and their concepts, after all the unanticipated side-trips into negotiating variables again. However, this kind of discussion shows the connection-making power of discourse when students perceive new experiences as informing them about prior understandings.

Should I just have declared one formula as the officially correct one? What if Mrs. Larkin or I had told them that textbooks which gave the formula as $P = 4s$ for squares and $P = 2L + 2W$ for rectangles were correct, or were redundant, or were communicating only the idea, or were communicating the idea and specifying the officially accepted variables for those concepts? What if we looked up the choice made for variables on the formula sheet provided to students who are writing provincial mathematics exams? How would they respond, when knowledge-authorities (texts, tests, a teacher, a teacher-researcher, and their recollections of former teachers) came into conflict? It might well have been an issue for some students at least about what counts as authority in math class, rather than the theoretical discussion about understanding which I would have assumed was occurring. Perhaps it was for the best that at the time I did not engage directly with the issue. Things were complicated enough. As the conversation closed, the label and formula *anttripOS: = 4 x sideOS:* was accepted as symbolizing the relationship for everyone's original square. The incident makes apparent the differences in students' understandings of what they take as purely mathematical ideas such as perimeter and common-sense ideas such as the invented ant-trip. This illustrates the value of teacher-led discourse which allows students to connect formal and personal mathematical ideas.. Another example of a common-sense idea available within the rectangles family context follows. It also provides an opportunity for students to connect common-sense mathematics with formal mathematical ideas.

Multiple relations, multiple variables, and multiple expressions

When more than one concept is being developed concurrently, the conversations become more complicated for the students, but comparisons among concepts become more likely. Deciding "how many irons to have in the fire" is a complicated pedagogic choice. Generally the students like to focus their attention on one specific element until it is resolved, but they may not recognize similarities and differences (make general judgments) when dealing with elements one at a time. In the above transcripts I chose to leave undeveloped a student's suggestion to count the squares of all different sizes, a relationship that is interesting in its own regard and one that offers some comparative value alongside counting the single squares (area). That is probably a good choice with most groups, given the complexity of the idea.

However, there is an idea closely related to the ant-trip which is quite simple to imagine and to count. If the Original Square (Parent Rectangle) was the floor plan for an office complex, how many offices would have windows? Although both this idea and ant-trip are perimeter-related, one of these ideas counts squares, while the other is a linear concept. I have been pleasantly surprised with students suddenly recognizing the fundamental difference between length units and area units when faced with this combination. Within the context of pre-formal algebra, however, the first value of the additional concept is to provide more practice at forming algebraic expressions. It can provide students with more practice at speaking and writing specifically (including with numerical expressions) about their Original Square and then speaking and writing generally (including algebraic expressions) in ways that describe all the groups' Original Squares.

Students are always surprised that the same idea can be counted so many ways. When this awareness can come from a large-group discussion, the students benefit greatly from negotiating their understandings with others. It is not a matter of looking at multiple answers to see which one is right, or best, but rather to look at them to see a concept in more than one way. Specifically, in a 14 by 14 grid-square, there are 48 outside squares, or offices with windows. That can be counted as:

$(14 \times 4 - 4)$,

(13×4) or $((14-1) \times 4)$,

$(14 + 14 + 12 + 12)$, or

$(14 \times 14 - 12 \times 12)$ minisquares, depending on how the counting of squares is visualized. The first expression shows that if you take the number of offices along one side and multiply by four (as with perimeter), you will have to subtract the four corner offices, because they have been counted twice. The second shows that the number of offices along one edge, with one corner office removed so it doesn't get counted twice, creates four equal sections of offices. The third expression describes the procedure of counting the complete top row and bottom row, and then the two sides (not counting the corners which were already done). Finally is all the grid-squares or offices (area), reduced by the inner grid-squares which have no windows. Each of these methods could be developed with different values if using the Original Square (Parent Rectangle) from the appendix.

If we wish to be describing any Original Square rather than a single case, we must replace each 14 with a variable such as *edgeOS:*. We can also coin a variable for the number of offices with windows, or equivalently the number of squares along the outside edge. This results in these expressions:

$outsquaresOS: = edgeOS: + edgeOS: + (edgeOS:-2) + (edgeOS: -2)$

$outsquaresOS: = (edgeOS: -1) \times 4$

$outsquaresOS: = edgeOS: \times 4 - 4$

$outsquaresOS: = (edgeOS:)^2 - - (edgeOS:-2)^2$

How do the students perceive all of these expressions as correct? At the time the students in this study were exploring these relationships and expressions, they had insufficient formal algebra to perceive arithmetically that each formula is equivalent. For convenience allow the expressions to be converted to more standard algebra by replacing "*edgeOS:*" with y. Formal arithmetic is much easier with simpler symbols, although of course the contextual richness disappears. The first three expressions above become $y + y$

+ $(y\text{-}2)$ + $(y\text{-}2)$, $4y - 4$, and $4(y\text{-}1)$, whose equivalence is simple to show with algebraic arithmetic. It is likely more of a challenge to see why $y^2 - (y\text{-}2)^2$ would be equivalent to the others, yet when expanded the y^2 terms disappear, and $-(-4y + 4)$ comes out.

The only arithmetic with variables which the students in this study had done prior to this instruction was some isolated grid arithmetic in support of the *areaA:* idea. Yet they do not need to refer (defer) to a mathematical authority (the teacher, the researcher, the text) to understand that each of these algebraic expressions are correct and equivalent: they can recognize its correctness within the context provided. The students could do so with simple arithmetic with any numerical value for *edgeOS:*. However it can also be done without referring to a particular square by understanding the counting system which the formula summarizes. This can be rich large-group conversation space, and should be anchored down by small-group conversations and individual writing after the ideas are broached in the large group.

The algebraic expressions emerge sensibly from activities upon objects within sensible contexts. Then the expressions themselves can be noticed as objects to be investigated (Sfard & Linchevski, 1994). Actions on those algebraic expressions can make their form and structure sensible as well. Although this study does not document the extent to which the students noticed the algebraic expressions which they had developed for self-expression as objects to be studied, the examples show the possibility of pre-formal algebraic experiences providing students with a foundation for viewing algebra's formal systems as both useful and sensible.

A further layer of relationships is available with the offices-with-windows expressions compared to the ant-trip expressions. Generally, they are both length-related and both are about four times the length of an edge. More specifically, for any rectangle *anttripOS:* is always four more than *outsquaresOS:* . Perhaps this is a problem with rich and natural contexts: there are enough relationships worthy of developing to overwhelm any classroom development. What is the pedagogic balance between sufficient complexity to provide some interconnectedness among relationships and the expressions that describe them and boundaries which help keep students from wandering without purpose or getting lost? Within the three periods discussed here, I stayed with area and perimeter as the two key relationships for each class of rectangle. *OutsquaresOS:* emerged as a label later when students pursued the idea as a review question.

More rectangles

In the next portion of transcript, I introduce the other members of the rectangles family to the students. These are the rectangles in Appendix D of a color different from the Original Square (Parent Rectangle). However, in the classroom I only provided each group with their Original Square, and this portion of transcript was my invitation to them to make related rectangles.

MrM: How did I make this one? [holding up Rectangle A in front of the Original Square]
S: Added one.
MrM: I added one row.
Ma: [Martin] You added one and subtracted one.
MrM: I added one and I subtracted one, well that shouldn't change it. Martin says I added a row and I subtracted a row, and I said that shouldn't change it.
Rb: [Ruby] You changed the shape but not the amount in it.
Hl: [Hilde] That's right.
Cl: [Colleen] Yeah, it will still be the same,
Hl: Number of squares.
Cl: Just in a different shape. Yeah.
Sv: [Silver] Right, right. See that line right there, it goes right there.
MrM: That red line's going to fill in that blue line.
Cl: Yeah.
Hl: Almost.
MrM: I wish I had squares on here to check you guys.

At this point, the discussion turned to other examples of rectangles which in size were somewhat close to the Parent, and each group of students were encouraged to make their own rectangles, "closely related to the Parent." Soon the room was over-run with lengths and widths and areas and ant-trips of many different rectangles. For any rectangle, the students could count the length and width, and then either count or calculate related ideas from the length and width. These related ideas could include the number of minisquares, the distance an ant would walk if walking around the rectangle, and the number of offices with windows. One thing became apparent quickly: we had a wealth of concepts to which values could be assigned. We risked being overwhelmed by variable labels, many of which each group had coined for their own convenience and adopted by mutual use within the group, only to find now that other groups had their own idiosyncratic language. This could have provided a glut of practice at negotiating common variable names, or we could have consumed considerable time stating particular cases of the general formulas for area and perimeter of rectangles. However, the value of all the rectangles and their various values lies in the opportunity to compare the values of two rectangles algebraically. This can generate equivalent but visibly different algebraic expressions for the same concept, similar to the comparing done above with the four expressions for *outsquaresOS:*. Here is an example.

When holding Rectangle A (or any member of the Rectangles Family), its area can be imagined as the product of its two sides and that can be confirmed by counting the squares. However, by juxtaposing it with the Original Square, Rectangle A's area can be imagined as equal to the area of the Original Square, give or take a specific amount. When the area (for example) of a rectangle can be viewed concurrently as the product of its length and width in one sense, and as equal to the Original Square's area with a slight arithmetic adjustment, then at least two different expressions for the rectangle's area can be derived. Why is $18 \times 16 = 17 \times 17 - 1$? If the values are all said in terms of the length of the Original Square, why is $(17 + 1)(17 - 1) = 17 \times 17 - 1$? The validity of each algebraic expression can be determined by the students through relating them to the rectangles and the counting procedures which relate. However, the ways which the two expressions relate arithmetically will not be as apparent to students who have not been schooled in algebraic arithmetic. In other words, students can form reasons for learning the arithmetic of algebraic expressions as they inspect the equivalence of two algebraic expressions for the same value.

To encourage the generation of multiple expressions for the same value, I insisted that within each group's family of rectangles the counting method(s) always be related to the dimensions and other values of the Original Square (dimensions of *sos:*, area of $(sos:)^2$, ant-trip of $4(sos:)$). Using other variables with arithmetic to label concepts was somewhat counter-intuitive for the students: why bother calling the length and width of Rectangle A $(sos: - 1)$ and $(sos: + 1)$, instead of just making new labels, such as *lengthA:* and *widthA:*? It may seem like referring to your aunt as your dad's sister, a second-order specification which is seldom worth the additional complexity. The long-range educational purpose of my making this requirement wasn't clear to the students, but again this is a use of a math teacher's authority that they understand too well. I did not stop to explain: we just agreed to carry on, on my recommendation that we would be able to find more powerful expressions this way, and we wouldn't have as many concepts to keep sorting out within and among groups. Of course on one level the purpose is to provide the students with algebraic expressions with one variable, so that they can learn about those expressions. However, within the students' inquiry the purpose is to be able to speak generally about relationships for all rectangles of a given kind. For example, for Rectangle A, where the length has been increased by one and the width decreased by one, the area is always one less than for the Original Square. How can that be? How could $(sos: + 1)(sos: - 1) = (sos:)^2 - 1$?

Other than brief interludes when groups were making their rectangles families or considering their own Original Square and Rectangle A, the first two periods were teacher-centred. The first period was sufficient time for students to develop relationships for their Original Square and for each group to make complete Rectangles Families. I had hoped that in the second period there the students would begin their group inquiries by midperiod, but some students found the comparisons from Rectangle A back to the Original Square to be confusing, and they demanded that I make the comparisons more clear. Rather than a quick overview of the relationships involved with *areaA:* and *anttripA:* described in terms of the Original Square's dimensions to be developed by students in small groups, we dealt with the details of the relationships through a teacher-centred approach. I was able to pose and outline the first inquiry for the next day, the small-group investigation of the same relationships for Rectangle B. However, I had not provided any chance for the students to try relationship-finding on their own.

My impression at the time was that the class as a whole felt good about the lesson. Clearly the ideas were complicated, but I had been careful to say that they weren't expected to "get it" by just following my one example. Generally the students were bewildered but intrigued with the inquiry they faced. The rectangles themselves provided some confidence by providing a way for them to hold concepts in their hands and confirm ideas about relationships by counting specific values. They were less confident to varying degrees about expressing the concepts and the relationships among them algebraically. As I was to discover in the lesson that followed, however, such a mixture of confidence and ambiguity following an introductory math lesson was new to many of the students. They were used to teachers' lessons showing them exactly what the sequential steps for their task was, and they had not been provided with that. They had followed the teacher-led development generally but were not certain at all of the details. This was as I had intended: their own inquiry with each rectangle in their family would provide multiple opportunities for students in their groups to build an understanding of the relationships involved and the role of algebraic expressions in describing them. I had stated that intention clearly, but stating an intention and convincing students that they will succeed with it are two separate tasks.

The focus now shifts from the curriculum activity as viewed from the designer-teacher's point of view to the experiences of the students. The interactions of three different groups of students on the third day will be interpreted, one at a time. These multiple opportunities may provide readers with opportunities to add detail to their understandings of the mathematical and instructional complexity of the rectangles families activity. More important, these opportunities will illustrate how this complexity interacted with and brought forward the students' senses of mathematics and how it should be taught and learned.

Relating (interacting and reacting) to rectangles

It is the third day of the strand. The students are in their groups working with their sets of rectangles. As a class, we have agreed to label rectangles "Rectangle A", "Rectangle B", and so on, so that everyone has a common label for the rectangles which have the same relationship with the Original Square. You have already dealt with Rectangle A, which is one row more in one dimension and one row less in the other. Rectangle B is one row less in both directions, and Rectangle C is two rows less in both directions. The students had rectangles D, E, and F, but they are not discussed in this document. The relationship of every rectangle's dimensions can be stated in comparison to the Original Square's single dimension numerically (the reader's Rectangle A is *(17 - 1)* by *(17 + 1)*), but that means each group's expression will be particular to its rectangle only. It can also be expressed in comparison to a variable name for the Original Square's dimension (*(sos: - 1)* by *(sos: + 1)*), which then will be true for all groups' Rectangle A.

Along with some review sheets for the previous algebra activities, I have provided each student with a simple sheet for recording their findings from the Rectangles Family. (See Appendix E.) The students are expected to develop two kinds of expressions for the area and perimeter of each rectangle. "Specific" expressions should use the numerical values for the dimensions of each rectangle. "General" expressions should use variable names for the dimensions of the rectangle based on each dimension's relationship to the dimension of the Original Square, *sos:*. As well as deriving these expressions, students should be able to explain the sense of every expression they develop by referring to the rectangle in question. Ideally, they should also notice relationships between equivalent expressions for the same concept. It is a worthy goal, but the educational rewards of this activity depend on the richness of exploiting the multiple expressions available for each concept and the interconnectedness of them, rather than dealing with one element at a time in sequence. It is dependent on experience-building, with understanding being derived from the experience of finding and expressing the relationships and discussing their meanings. Certainly the activity is too complicated generally to make a received explanation satisfying, before a student (or a reader) begins to explore.

The group that is the next focus of attention had no difficulty with the complexity or ambiguity of the task. The four students have accepted my suggestion that every different answer to a question (i.e., every expression for a given concept) is worth a separate checkmark. This transitional sense of purpose is intended to fill the void until the activity makes sense on its own behalf. Because they felt comfortable with the examples developed the previous day (the Original Square, Rectangle A, and (briefly) Rectangle B), they began work on Rectangle C, which for them is 14 by 14, in comparison to their Original Square of 16 by 16. The readers' Rectangle C is also 2 cm less than their 17 by 17 Original Square, as would every group's Rectangle C.

Sv: [Silver] Okay Piece C, let's do that one. Fourteen by fourteen. So you have to do it for individual examples, and for,
Mo: [Monuel] *sos:* right?
Sv: General. Like this. Everybody got this chart down?
Mo: Yeah, actually, I do.
Sv: Thanks.
Hl: [Hilde] Oh, could I borrow a ruler?
Mo: Sure.
Hl: Oh actually I have one. I come prepared never mind.
Mo: *sos:*. Equals. Oh I don't have to do this, right. So it's Area
Mr: [Maria] You guys. Wouldn't this be similar to the fifteen by fifteen? It's just two off instead.
Mo: So *sos: minus two*? That's for general.
Sv: We should do individual first though right?
Mo: Okay, so how does this go?
Sv: It's fourteen by fourteen. .
Hl: So *areaC:* is *14 times 14* right?

Sv: That equals, .
Mr: That equals 196.
Sv: Okay.
Hl: Is there any other way to write it?
Mo: Yeah , *sixteen plus negative two, times sixteen plus negative two.*
Sv: *Sixteen plus negative two* comma,
Mo: *Times sixteen plus negative two.*
Mr: Okay.
Mo: See we get more checkmarks. Let's try another way. *Sixteen minus two,*
Sv: *Squared.*
Mr: *Sixteen squared, minus,*
Mo: I got that.
Sv: Oh yeah, okay, good point.
Mo: Oh yes, four checkmarks! Okay! And now Anttrip.
Sv: Anttrip? Aren't you going to do general?

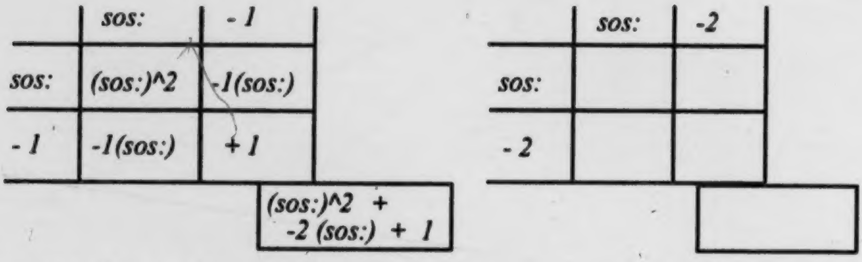
Autonomy and interdependence

These four students have not been formally introduced. This group demonstrated what small-group learning can be like: interactive with but relatively autonomous of the teacher, and deeply interdependent in regard to each other. Monuel (Mo:) was indomitable, tending to make big leaps intuitively, very much enjoying this style of learning. Hilde (Hl:), who earlier in this chapter was the arbitrator of variable names, and Silver (Sv:) were more disciplined about progressing logically through each task. Hilde took it on herself to ensure that the group members felt good. Silver, who had expressed that she would like to be a teacher, took it on herself to ensure that every group member understood. The group relied on Maria (Mr:) to make sure they made sense of the ideas they developed, and to attend to essential details as they progressed. Maria carried less confidence, and as a result did not sell her good ideas as aggressively to the group as they deserved. An example of that is in the previous transcript, where Maria began to suggest that they expand $(16 -2)(16 +2)$ into separate terms, but it didn't stick. Yet, in the transcript below, she succeeds in getting the group to follow through on that idea with the general expressions. They were as effective a group as these transcripts make them appear.

As we pick up the conversation where we left off, the group attempts to describe *areaC:* generally, in other words in terms of *sos:*. At this point, their answers would describe any group's Rectangle C as well as their own. Notice that the students have an intuitive understanding that there is a way to express *areac:* in comparison to $sos{:}^{2}$, but have to struggle to find it. They refer back to their notes from the day before, where they see that for *areaA:* the whole-class lesson had derived *(sos: - 1)(sos: + 1)* and its expansion, $(sos{:})^{2} - 1$; similarly, for *areaB:* they had in their notes the expressions, *(sos: - 1)(sos: - 1)* and its expansion, $(sos{:})^{2} - 2\,(sos{:}) + 1$. As they struggle to determine what the equivalent expansions would be for Rectangle C, they try to reproduce the grid arithmetic (what Silver refers to as "tic tac toe" and Monuel refers to as "X and O's") used the day before. They get the four cells filled in successfully, but only through their shared persistence do they make sense of what to do with the two *- 2 (sos:)* terms. Because they do not know what constitutes correct arithmetic procedure with the grid and with variables, they can only check their results as they do: in part by comparing to their primitive understanding of arithmetic with negatives, and by referring back to the rectangles being described. As a result, they seem to have a better understanding of the grid arithmetic, and, hopefully, of algebra as a system for expressing equivalence.

Mo: [Monuel] Okay, we'll go to general. So it's *areaC:*,
Hl: [Hilde] *sos: minus two, times sos: minus two.*
Mo: *sos: minus two, squared.*
Mr: [Maria] *sos: minus two, times sos: minus two*, right? Okay.
Hl: *sos: minus two, squared.* Okay, let's do one of these little things.
Mo: Is this right, Hilde? You can only have two of them, for this one, right?
Hl: No, there's another one, but I can't figure it out.
Mr: You know what he did, these two more. . What is this one here? He goes for *areaA:*. He said it was like *sos: squared minus one*. Can we do that, *sos: squared, minus, three*? What would that be?
Sv: [Silver] Look at B.
Mr: Look at B? Okay.
Sv: Hang on. Look at *AreaB:*. Are you on *AreaB:*? You never wrote it down. Okay, I've got *AreaB:* here.
Mr: Yeah, I have it.
Sv: *AreaB:* is *(sos: - 1), times (sos: - 1)*,.
Mr: He did *sos: squared, minus two*,
Sv: *sos: squared minus sos: minus sos:*, okay, let's just do it this way so we can see what's going on. [See Figure 4-1, part 1.]
Mo: That *sos: squared minus two*? That's for how much was taken off. What's

the difference?



	sos:	*- 1*
sos:	*(sos:)^2*	*-1(sos:)*
- 1	*-1(sos:)*	*+ 1*

(sos:)^2 + -2 (sos:) + 1

	sos:	*-2*
sos:		
- 2		

Figure 4-1. The area of Rectangle B compared to Original Square, and the area of Rectangle C.

Mr: Yeah, he's right. . No, .
Sv: Okay, oh I see what I did here.
Mo: What did you do?
Sv: Well let's just do it like this first.
Mo: There is no other way.
Sv: No no there is another way. [holds Rectangle B up to the Original Square, to compare their sizes.] Try out this like this. [points back to the Figure 4-1 grid.]
Hl: Tic tac toe?
Sv: Yeah like this. Just like this. . Okay let's do it out like this.
Mr: Okay.
Sv: Okay, so you go *sos:*,
Mo: Are you doing this?
Sv: That's for this.
Mo: Oh yeah.
Sv: You see what I am doing?
Mo: And then you try and find the answer? It's *sos:* squared,
Sv: *Squared,*
Hl: *Plus two sos:,*
Mo: *Plus negative four*. Yeah! There it is! *sos: squared plus negative four*. Right?
Mr: No.
Sv: Just a second.
Mo: You sure?
Mr: Can't be. . Because you don't go *negative sos: plus negative sos:*, right?

Okay, I don't know what I'm doing.

Mo: Hilde's doing this right.

Hl: I don't even know what I'm doing, guys. .

Mo: *sos: plus a negative two*?

Mr: Yeah. Those two don't cancel out each other do they? Oh, this is timesing, yeah they do.

Sv: No they don't. They're added, so they're still negative.

Mr: *sos: squared.*

With dynamic transcripts, it is always difficult to interrupt. Here the students are truly stuck (J. Mason, Burton, Stacey, 1985) but actively pursuing resolutions of their difficulties. The students express their perspectives, each one different. Hilde is filling in the four cells of the grid, uncertain that she is correct. Monuel has leapt to a simple answer, imitating the answer for rectangle A, which was $(sos:)^2 - 1$, because two of the four cells from the grid multiplying were opposites and disappeared. He knows that Rectangle C is even smaller than Rectangle A, so his answer feels right to him. Maria is looking more closely as Hilde fills in her cells, and is correct to doubt that two of the four cells will "cancel out". However, her confidence is undermined by her shaky sense of when to multiply (each of the four cells is the product of the two corresponding terms on the outer edges) and when to add (the final answer is the sum of the four cells), and all the negatives don't help! Silver only tentatively proposed that they look more closely at the rectangle in comparison to the Parent, and since her suggestion has been temporarily abandoned, the students are investigating the grid arithmetic as an algorithm which they don't yet understand (Bruner, 1960/1995). As we rejoin the flow without missing any conversation, it gets a little muddled for us as the students talk about different cells without indicating which they are referring to. In fact, it's a little muddled for them, too.

Hl: *sos: squared*, and these are minus,

Mr: So negative, would it be *negative sos:*,

Hl: To the power of two? No?

Sv: Let's do Anttrip.

Mo: Yes, I agree.

Mr: Perimeter, right?

Mo: A negative times a negative is a positive *sos:*.

Sv: No, shouldn't it be *negative four sos: plus two*, never mind.

Mo: It's *four sos:, six sos:*, no it's not.

Sv: Yeah, would that add up too?

Mo: So it's *sos: squared, plus sos: times four,*

Sv: Mm-hmm, *plus,*

Mo: *Plus four!*

Sv: This is what I had. Where did you get *four times sos:,* Monuel?

Hl: Not *four times sos:,* just *four.*

Sv: No, shouldn't it be *negative, negative four.*

Mo: What, *two sos:*es, *two negative sos:es,*

Sv: Oh, this is hard.

Mo: Is a positive.

Sv: Okay.

Mo: So is this right, I'm going to write it out. .

Mr: Do you get it?

Sv: Hmm?

Mr: How did you get that?

Hl: And don't forget the *plus four*, too.

Mr: *sos: to the power of two, plus, four times sos:* ?

Sv: Right.

Mr: How'd you get that?

Sv: *Negative two sos: times negative two sos: is four sos:.*

Hl: Okay, let's see if it works. [picks up Rectangle C and the Original Square]

Sv: Cause *negative two times negative two is four,*

Mo: I'll try it Hilde, you don't have to use your, xx.

Sv: That's *four, four sos:,* and this is where you get *four sos:.*

They've made great progress, but they are going to make even more. This extended inquiry is remarkable, because they are investigating the grid arithmetic tool itself, as much as they are the algebra and the rectangles. None of the elements or tools at their disposal are familiar or self-evident, but they have a sense that they can figure out all of them if they are persistent. Here, the group of four have achieved consensus on the four cells: $(sos:)^2$, two cells of $-2(sos:)$, and 4. They aren't sure what to do with the negative aspect of the middle cells. It is time for them to look at their rectangles. Positioning Rectangle C in front of the Original Square will make visible the four rows of *sos:* that must be removed from the Original Square to make Rectangle C. Also visible are the four corner squares that get subtracted twice by that process and must be added on. As we rejoin the flow, the group is looking at their two rectangles, too. But Monuel switches to specific values, and does the arithmetic with 16 replacing *sos:*, instead of thinking in general terms! Just as with the students themselves, readers need not understand each turn. The students keep their goal in view while listening to each other: to decide what terms are positive and negative (added or subtracted) , to complete the answer where I have left blanks.

$$areaB: = (sos: - 2)^2 = (sos:)^2 \_\_\_\_ 4(sos:) \_\_\_\_ 4$$

Mo: *Sixteen squared, times, equals*,

Mr: So then it's

Hl: No that's not going to work. [She sees that 256 is a long way from 196.]

Mo: So it should be negative.

Sv: Yeah, because you're adding the two fours, not timesing them.

Hl: Yeah, that's what I thought.

Mo: Yeah, that's right, okay.

Sv: So it's *negative four*.

Mo: So it's negative, but then it will still be positive.

Sv: Cause you're adding, no, negative four, times a positive number,

Mo: Oh, so it's *negative four times*, yeah, okay.

Sv: So it's *negative four times*,

Mo: So *four times sixteen*,

Mr: So why'd you *plus four* at the end?

Sv: Because there's a four right here.

Mr: Ahh. Good point. .

Mo: It works!

Sv: Does it?

Mo: Yeah.

Sv: So this is another way you can write it.

Mo: Yes, four checkmarks!

Sv: You have four, I only have three. Oh yeah, what about that one? [$(sos: - 2)(sos: - 2)$] Oh okay.

Mo: There we go. Okay. Now we're on Anttrip. Holy. That was good. I like this. I'm going to do this on all of mine, this Xs and Os.

Surely this is how small-group inquiry is supposed to occur. As a group these four students have done more than any of them could have done on their own. Not only have they obtained a satisfying answer, but they have experienced success within a difficult learning process. They had to draw together elements from a range of partially understood concepts, work with those elements, and check their outcome against their understandings of what was correct. Above all else, their willingness to deal with ambiguity and complexity while searching persistently for sensible connections has paid off (Clarke, 1995). Grid arithmetic can align with the algebraic statements that describe the relationship between two rectangles, and the algebraic statements can align with their numerical equivalents in a particular case. They do not have sufficient practice to be good with all of "this", and they are a long way from being able to say how or why all of this "works", but they understand what all of this is about.

Yet just because their understanding, as represented by their four-checkmark answers, was developed by the group, can we say that it belongs to the group? On close inspection, it is clear that all four provided vital aspects of the flow, although at times one person's profile is higher and at times only two or three of the group members are speaking. In fact, only Monuel has enthusiastically claimed to understand. Perhaps it is better to say that the coming to understand was situated in the group, but that we are not sure what understanding each student has constructed through this interaction. "Ideas ... are acquired socially, through the use of language (i.e., sign systems). The testing out of these anticipations, however, is a personal, experiential process." (Prawat, 1995, pp. 17-18). It would be a good idea to be able to check for personal understandings, to have group members say what it is they understand. As we join the conversation again, Hilde thinks that would be a good idea, too.

Sv: There okay. Is everybody ready, Maria, do you need understanding help?
Mr: Yeah, just with this here.
Sv: Okay, did you do this?
Mr: Yeah.
Sv: Okay. When you times this *sos: squared*, right?
Mr: Yeah.
Sv: I just wrote it down. Now this is *negative two sos:*, it should have brackets. And *negative two sos:*,
Mr: Yeah.
Sv: *Plus four*. Is that right? So what we did is we added *negative two sos:*, *plus negative two sos:*, right we did this,
Mr: Yeah, okay.
Sv: And we got *negative four sos:*, right?

Mr: Uh-huh. Okay, yeah.

Sv: And when you multiply *negative four sos:*, it's *negative sixty four plus four* is *negative sixty*. *sos: squared* is, *256*. *256 minus 60* is *196*, so it works. So it works.

Mr: Yeah.

Sv: Do you got it?

Mr: Okay.

Sv: Did you write it down?

Mr: Yeah.

Sv: Okay. Anttrip. Do you want Anttrip, guys, is everyone on Anttrip.

Mo: Okay, we're on *anttripB:*, specific, right?

For the sake of brevity I will summarize the group's success with *anttripB:*. The group disagrees about whether *(16 - 2) x 4* and *(16 + -2) x 4* are distinct enough to count as separate answers but they agree that *(16 x 4) - (2 x 4)* is distinct and makes sense. When they switch to talking generally, they arrive at *4(sos: - 2)* and *4(sos:) - 8*. they can explain why they are both sensible in terms of ways to count the ant-trip and why they are equivalent. When I arrive, they feel confident and competent and ready to go on. We discuss the struggle they had figuring out what to do with the inside pieces of the grid arithmetic, and agree that it's hard to decide what makes two answers for the same thing different enough to be worth having both. Finally, Maria asks how I could successfully test for this sort of thing, with questions to be done individually: "Some of us have a better understanding of algebra when we work in groups and we discuss it, rather than all of a sudden you give us a question and we're sitting here on our own, you know." Her words suggest what it means to synthesize an understanding with others: "But you know why, Mr. Mason, because each of us, we all have an understanding of like one part of it, then when you put it all together, then you understand the whole thing."

These students are remarkably capable at formulating understandings through cooperative group inquiry. They have provided a wonderful example of the construction of understandings which Chapter 3 described as a possibility. However, it would not be fair or responsible to assume that all students could have been as successful at the construction of understandings as these students were. That raises a key question: Does constructivism, small-group inquiry, and problem-based pedagogy mean simply that a different set of students will be considered the most capable, or perhaps that the same students will be considered the most capable but for different reasons (Mathematical

Sciences Education Board & National Research Council, 1990; National Research Council, 1989; National Council Teachers of Mathematics, 1995)? I believe it can be much more. I believe that within this pedagogy is a frame for helping all students, not just those who are predisposed and skillful at this kind of learning process, like Monuel, Maria, Hilde and Silver. That is why I feel obliged to put beside the above example the experiences of other groups. In the next chapter, you will meet two other groups of capable, conscientious students, but these students have a very different experience with the Rectangles families inquiry space.

Chapter 5. Struggling with Rectangles Families

This chapter provides our first experiences with the students from two more of the small groups of learners, all of whom will also play prominent roles in future chapters as well. The second half of the chapter will introduce Ivan, Wai, and Martin, but first it is Kelsey, Ruby, and Colleen's turn. Kelsey, Ruby and Colleen all had a very different relationship with mathematics as a school subject compared to the students in the previous chapter. As a consequence, they had a very different experience with the Rectangles Families inquiry activity which they had been told to do. Kelsey, Ruby, and Colleen were bright, eager to succeed, and diligent. As a small group the three students worked very well together, having similar ideas of how to succeed at math. For me, all three students were vital barometers of the pressure which built up at times, in that they were vocal and cooperative in requesting that the teacher explain in more detail or change procedures when they thought it necessary.

It was partially in response to their expressed concerns that on the second day with the Rectangles Families that I provided more than an overview of the kind of exploring I expected. I had intended for an overview of what to do with Rectangle A to provide only a general sense or outline of the inquiry with Rectangle A before the groups explored the possibilities, but as Kelsey and others insisted at each step that I explain that step fully, it had dragged out to occupy the whole period. At the end of the day, frustrated with how slowly we went and confused about why some students had tried to write down every detail, I asked the homework question, "What should you write down, and why?" For these three students, the second day's complicated overview had increased their anxiety, not decreased it. Mrs. Larkin chose to join Kelsey, Ruby, and Colleen today, and both the conversation and the pedagogy is richer as a result.

Teacher expectations and student roles

> Roles may be reified in the same manner as institutions. The sector of self-consciousness that has been objectified in the role is then also apprehended as an inevitable fate, for which the individual may disclaim responsibility. The paradigmatic formula for this kind of reification is the statement, "I have no choice in the matter, I have to act this way because of my position" (Berger & Luckmann, 1967, p. 91).

> "Constructing" and "introspecting" may not represent expectations for behavior that students in a particular district, school, or classroom are accustomed to meeting. If, instead, these students have routinely viewed appropriate actions associated with learning as memorizing information or as replicating problem solutions, then an important discrepancy exists between adults' reform expectations and students' daily enactment of the role (Corbett & Wilson, 1995, p. 13).

> There is hurt in learning, and it is difficult to persuade someone to hurt himself. ... It is especially hard for adolescents, whose vulnerability and inexperience are attenuated. Getting them to pursue this often lacerating process of exposing that inexperience, and the errors it reaps, is a subtle, delicate business (Sizer, 1984, p. 159).

The transcript below shows Ruby, Kelsey, and Colleen beginning the small-group portion of the inquiry. From the beginning the students make their frustration apparent. They are not used to feeling confused before beginning an activity and it adds to their apprehension that other groups around them are getting started without significant anxiety. Mrs. Larkin's calm presence gives them someone to express themselves to, as well as a level head to help them work through their difficulties. Typed transcripts have a difficult time capturing emotional nuances. Rather than provide my sense of the statements said with sarcasm, frustration, or any other special tone, I have simply used double exclamation marks to indicate that there was intensity in the statement. The expressions of Kelsey and Colleen marked in that way are sometimes angrily said and sometimes sarcastic, but even their harsher words are usually said with a tone appropriate for excellent students.

Kl: [Kelsey] What are we supposed to do with this?

Cl: [Colleen] Yeah, he just said, Take out Rectangles, Painted Cubes, and Handshakes, Odd Jobs and Passways. Then what are we supposed to do with that?

Rb: [Ruby] What are we supposed to do with them?

MsL: [Mrs. Larkin] Take out the notes you did yesterday.

Cl: Well I know!!

MsL: Take out the notes and read what you did yesterday. And start from there. You have to start somewhere.

Kl: Well duh. But that part's done.

MsL: Are you ready?

Kl: No, but my homework is on the stuff that I have to record.

Cl: Yeah!!

Rb: [whispered] He's coming.

Cl: Oh Good!!

MrM: [Mr. Mason] You need your work from yesterday's class. You're frustrated today, are you?

Cl: Yeah, very much like that.

MrM: I agree with what you wrote by the way. Remember the stuff we talked about yesterday, because I deliberately went too fast for you to understand it all yesterday.

Cl: Yeah, very much too fast!!

MrM: Too fast for that choice, to "write down everything." [quoting from Colleen's homework. Mr. Mason leaves.]

MsL: Okay, get out your stuff from yesterday and let's start going over it.

Cl: Can I borrow a piece of paper? I don't have any.

MsL: Where do we go from there?

Kl: I don't know.

Cl: I didn't get lots of stuff done on this!!

MsL: Well let's go where you started. What's the first thing we were talking about? The first thing you did. .

Cl: Umm, Kelsey, can you turn your desk around so we can look at your lovely face?

Kl: Well you don't want to!!

Cl: Yeah but you're talking to the chalkboard and I can't hear you.

MsL: Don't be crabby. You don't want to.

Kl: [laughs] Exactly. She's having, a very bad day.

MsL: She's so crabby, do we want her here?

Kl: Not really.

Rb: Why don't we put her in the closet till she calms down.

Cl: No she [herself] has some valuable input sometimes.

MsL: Maybe when she's angry she's growing.

Kl: [laughing] Whoo then I'm growing all the time!!

MsL: Okay then, what do we do?

Kl: Okay, well,

MsL: What do we do, start off with a square?

Kl: No we have to find the squares we did so far.

MsL: Let's do what we did yesterday first. What did we do first? You tell me.

Cl: Fourteen by fourteen. Or, that S O S. [In comparison to the readers' Original Square which measures 17 by 17 (Appendix D), this group's Original Square measures 14 by 14 and everyone else's in the class is different from both.]

MsL: Okay.

Cl: Where did that go?

Kl: Argh, I can never draw straight lines.

Cl: Well you don't need to, Kelly.

Kl: They're straight, they're just crooked.

Cl: I don't have my calculator. I lent it to someone.

MsL: Well couldn't you do it in those rectangles? [the grid arithmetic]

Cl: Yes. 196.

Mrs. Larkin has an insightful idea of course. The students need to go over what they did yesterday as a class and make sense of what they can do and look more closely at what confuses them. Kelsey, however, views the matter differently. The Original Square and the two rectangles that were done in class yesterday are done. She believes that to get done, it would be better to go on to other rectangles. She is not used to doing mathematics more than once to understand it.

As we rejoin the conversation where it left off, the students engage in the renaming of variables. This is not a trivial process: here the girls discuss the alternatives to using names which carry meaning, before agreeing to use a meaningful full-word variable of their own choice. I will return to discuss my sense of this, not in terms of algebraic understanding as in Chapter 4 but in terms of students' relationships with the authority of the subject matter. At the end of the chunk, the group arrives at valid answers for the area of the Original Square, both in general terms using their variable and in specific numerical terms. Readers who follow along with their Original Square will be using 17 and 289 where the students use 14 and 196 when dealing with specific values. The general forms will work for any Original Square, whether referring to the length of a side with *sos:* or *side:*. and with whatever labels are selected for the other countable and derived values.

MsL:: Who wants to write that out? Let's make a list of what we're doing here. If you want. If you already have it you don't have to.

Kl: Let's find the area first. which is *sos:* squared!!

MsL: Okay. You don't like *sos:* though. Change the words.

Cl: Which is?

Kl: Original, Square.

Cl: Side! *Side: squared.*

MsL: *Side: squared.* Okay, write down *side: squared.*

Kl: I hate having, Why does he use these quotes [she means the colon at the end of variables] at the end here?

MsL: So you always know that's where a number goes. So that xx,

Cl: Why don't we just use a number?

Kl: Or just *n*?

MsL: *N* can be very confusing. *N* can mean a whole bunch of different things. If you use a different word,

Rb: Well then why not just do a number sign? Like that thing [#]. It means number.

MsL: Okay, and is it going to mean the same number when we do Painted Cubes?

Cl: Well why don't we just use a different letter for each number?

Kl: This [full words with a colon] is easier than the number thing.

MsL: Okay then, *Side: squared*?

Cl: Okay, but, that would be general though.

MsL: That would be general.

Kl: That's our original.

Cl: I knew that!!

Kl: That's all the stuff from yesterday. So are we doing it all again?

MsL: No we're just going through it. Okay? And the specific would be?

Cl: *Fourteen times fourteen.*

Kl: Or *fourteen squared.* No I put my examples in.

MsL: What are you doing? This is not hard! She's so brilliant. She thought of a plan all on her own. That Colleen. Oh, are we lucky to have her in our group.

Kl: Yeah.

Cl: Okay so then specific would be, oh and this is all area right.

Kl: Well I understand the area part.

The above transcript includes a rich example of how negotiating variable names can include student mean-making about what variables are for. It also shows Colleen's frustration easing as she and her group-mates are able to agree on her choice for that variable. Does Colleen dislike the idea of complicated variable names or the idea of the class negotiating those names? Does she just want to control what is used for variable names? I think the issue for her is not about the variable names themselves. Rather, she is uncomfortable with the ambiguity which the negotiated variable names represent. Imagine what it must mean for a student who believes that math is predefined, that the math teacher's job is to present mathematics in a clear and simple way, and that the student's job is to practice the use of that mathematics until she is able to use it on command when the test comes. Suddenly there are different concepts with multiple names (lengths of sides or dimensions, numbers of mini-squares or area, ant-trips or perimeter) for each of the different rectangles in her group, and other groups' rectangles are different again. The formulas being learned appear to have none of the utility of basic formulas that $A = L x W$ has, so even the goals of the activity were ambiguous for Colleen. For her, the challenge of making sense of all these ideas wasn't what math class was supposed to be like, and the ambiguity of what variables to use and what they should mean is for the textbook or the teacher to specify. Colleen could not interpret what was frustrating her in detail such as this, however: she could only complain or do nothing. Too good a student to simply do nothing, by complaining she began to consider what frustrated her.

Following the above portion of transcript, the students discussed perimeter for the Original Square, and it went well. Ruby and Colleen helped Kelsey see that $14 + 14 + 14 + 14$ is the same as $14 x 4$. As the students realized that maybe it wasn't going to be too difficult, a listener might have assumed that everything was going well. As the transcript picks up the conversation, they succeed in expressing the relationship algebraically before switching to dealing with the next rectangle, when Colleen and Kelsey suddenly return to expressing their frustration from the day before. They still resent that the previous day's lesson had left some details unclear. Yet at the end of this portion, it is Colleen who tells Kelsey she has had enough complaining, and it's time to figure this math out.

Cl: No what would general be?
MsL: Sorry generally sorry.
Kl: *Side colon times four.*
Cl: *Side colon times four.*
Rb: Well that's not hard.
Kl: Well you did it yesterday. It was very hard.
Cl: Because we used all those different words and,

Kl: Yeah.
Cl: I didn't understand. Especially *anttripOS.*
MsL: All right. That's fair.
Kl: Okay, then I put, no, that's Piece A.
MsL: What was different? This is Original.
Kl: No Piece A is fifteen by thirteen.
MsL: Piece A. Do you want to do piece A first?
Kl: Yeah. Piece A first.
MsL: Okay, fifteen by thirteen that shouldn't be too difficult. *13 times 15.*
Kl: And I don't want to get confused. When he's up there he'll be talking about something totally different and I'll get confused.
MsL: Okay.
Cl: Kelsey. Calm down Kelsey. Chill. Okay Piece A. Okay that says *thirteen times fifteen,* right?
Rb: Right.
Cl: So, it would be,
Kl: Side times side.

I have interrupted just as Kelsey was starting to formulate a formula for area. Her starting point wasn't completely wrong: although *side: x side:* will only give the area of the Original Square, students the day before had expressed that this rectangle had one row taken off of one side and one row added on to the other side, so its area should be the same as the Original Square's area. Overlapping Rectangle A with the Original Square can make apparent both the sense of that thinking and its inaccuracy. However Kelsey's thinking wasn't about the equivalence of the two rectangle's areas. Instead, Kelsey was just stating that area is derived from length times width. In a sense, she was saying "side times side", not "*side: x side:*". It is a good starting point, and stating it helped her to see how to improve it. Her next sentence was, "Okay this is what you do. You go *side: minus one times side: plus one.*" The students were arriving at a consensus about the relationships and how to express them through their discourse, including using algebraic variables according to the conventions governing them. It was partly the reconstruction of the previous lesson as the students remember it and partly their analyzing of their own Rectangles Family. The discourse provided the process for them to be able to improve their imagery until it made complete sense.

Re-viewing the rectangles

> Disequilibrium [is] an essential catalyst for learning new skills and knowledge, for differentiation and integration (Chickering & Reisser, 1993, p. 1).

> This stance of mine often caused frustration for an individual learner, but I held firm and returned to individual support of that student's learning process and efforts. ... Losing their comfortable grounding was usually scary for students, but only when that happens can major learning and change occur (Dillon, 1995, pp. 195-196).

> Being challenged, encountering novelty, confronting one's misconceptions -- in short, building new and stronger understandings, -- typically involves bewilderment and frustration. However, if mathematics students can learn to recognize that the discomfort they experience is part of the process, they can also learn to tolerate it (Schifter & Fosnot, 1993, p. 11).

With Mrs. Larkin's encouragement, the students did the grid arithmetic for the area of Rectangle A, using *(14 - 1)* and *(14 + 1)* for the two sides (see Figure 5-1, part 1). Ruby recorded the arithmetic for the group in part because of Colleen's reluctance. "No. No, because I'm not very good at negative numbers. I suck at them. I do, I try, but I can never get it straight. I always put the wrong answer." There is no indication whether Colleen had memorized or understood her integer arithmetic the previous year but we could hypothesize based on her sense of how she believed math should be learned.

	14	+1
14	196	+14
-1	-14	-1

196 - 1 =
195

	side:	+ *1*
side:		
- *1*		

Figure 5-1. The area of Rectangle A compared to Original Square, first numerically and then algebraically.

The grid arithmetic is not an easy idea, but the group succeeds with the numerical version for Rectangle A. They do not discuss it when done except to acknowledge that it did provide the right answer. They are getting things done.

Below, as we rejoin the flow of conversation, the group is switching from doing the area for Rectangle A with numbers to doing it with variable names. It is during this experience that Kelsey expresses recognition of what *side:* means. When she and Colleen express in a bright flash the short-cut answer for multiplying *(side: - 1) (side: + 1)*, Mrs. Larkin slows them down, has them do the grid arithmetic (see Figure 5-1, part 2) and compare it to the numerical grid, and Colleen "gets it". It isn't easy to follow at first since it is difficult to know exactly what is written down at each step but I think the key moments [flagged with the initials KM] will come through. First Colleen recommends thinking about the numerical specifics while doing grid arithmetic with the algebraic form, marked below with [KM 1]. Quickly thereafter Kelsey realizes that the big idea with Rectangle A is to compare to the Original Square [KM 2], and that makes apparent what the class had been doing with "*side:*" all along [KM 3]. Following that they leap ahead in a way which Mrs. Larkin does not anticipate and a rich conversation results in which they get the mathematics sorted out.

MsL: So far so good?

Rb: Kind of.

Kl: It sort of goes,

Cl: Is this area right?

Rb: Side? No.

Kl: Let's try them out, this is empty.

Rb: Oh no.

MsL: This is where you need to make your little square again. That's why we did this. This is *Side:, minus one*, This is *side:*, [See Figure 5-1, part 2.]

Cl: Yeah so it's *side: minus one, times side: plus one.*

Kl: Don't tell us. When we have the Side here, we got, whoa,

Cl: Okay, so you just think *fourteen minus one*, so *side: minus one*,
[KM 1]

Kl: But how did these go fifteen and thirteen, how did they know what Side it's going to be?

Cl: Oh, look! Side is *fourteen minus one* is *thirteen*.

Kl: Oh it always goes to the original!
[KM 2]

Cl: Yeah!

Rb: Yeah!

Cl: *side: minus one*. And *plus one*.

MsL: And some people might have an eleven and some people might have a sixteen, okay. Because xx.

Kl: So *side:* will always be the Original Square number.

[KM 3]

MsL: Yeah.

Kl: Mm-hmm.

Rb: Oh no.

Cl: Oh so we have to do that with letters now?

MsL: Now do it with words. Side colon.

The Rectangle A from Appendix D could be used to proceed in parallel with the students' discussion of their Rectangle A. Numerically, the area values would be *16 x 18*, or (in comparison to the Original Square), *(17 - 1) (17 + 1)*. The numerical grid arithmetic for this expression would have the same structure shown in Figure 5-1 but different values. The algebraic grid arithmetic for *(side: - 1)(side: + 1)* would be identical to that in the transcript since in general terms both the readers' rectangle A and the group's have the same relationship to their respective Original Rectangle. In other words the area of anyone's Rectangle A is one less than the area of their Original Square. The transcript resumes where it last left off.

Kl: That could also be, what else could it be, *Side dot dot squared minus one*.

MsL: Right and then *Side dot dot squared plus one*.

Kl: No.

MsL: Because you were originally multiplying, Sure you are.

Kl: Mm-mm.

MsL: The thirteen is the *side: minus one*, the fifteen is the *side:*,

Kl: But lookit.

MsL: *Plus one*.

Kl: Yeah but okay, that's not what I'm saying.

Cl: Can we say it, *Side squared minus one*?

Kl: *Side squared*.

Cl: *Minus one*.

MsL: That may be your final answer, but I want you to do it here, first.

Cl: Okay.

Kl: So you go, *Side colon minus one,*

Rb: Oh you have to do the little rectangles [grid arithmetic]. You did it wrong.

Cl: Oh I did it wrong. I knew that. . This confuses me even more.

Kl: Oh, time out.

Cl: I don't understand these boxes [grid arithmetic] at all.

Rb: I don't understand these boxes.

Kl: Whatever I put in these two boxes, they have to cancel each other out, but what do I put?

MsL: So this is going to be *side: times side:*, or *side: squared.*

Kl: Yeah.

MsL: And this is *minus side:.*

Kl: *Minus side:.* And this is *positive side:*?

MsL: And this is *positive side:*, and this is *minus one.* Right?

Kl: No it's *negative one.*

Rb: Oh! Yeah. Well *negative one* is the same as *minus one.*

Kl: Yeah.

MsL: Okay, let's see how that is related to this. 196 is *side: squared. Minus a side:* , so you had *minus fourteen.*

Kl: Yeah.

MsL: *Plus a side:* .

Cl: Yeah.

MsL: *Plus minus one.*

Kl: *Plus minus one.*

MsL: Does this match?

Kl: Yes they do.

MsL: Okay, and what's the short cut answer, *side: squared minus one*?

Kl: Yup.

MsL: Would it be *side: squared minus one* here?

Kl: Yup.

MsL: There you go.

Cl: Huh?

Rb: So *side: squared,*

MsL: Match the Sides. This is *side:* . Fourteen is the *side:* .

Cl: Yeah.

MsL: So you got *side: squared* 196. *Minus a side:* , which we had before as *minus fourteen, plus the Side,* which was *fourteen* here, *minus the one.*

Cl: So it's,

MsL: So the shortcut is *196 minus one* because these two fourteens cancel each other.

Cl: Ohhh, okay. I get it.

MsL: So it's *side: squared minus one* generally.

Cl: I've got to write that down. I understand.

MsL: Does that make sense, are we making progress?

Cl: Okay.

Rb: Okay.

This is a positive moment. Not only do the students understand the finer details of the ways in which variables have been used to identify particular ideas so that they can be compared, but they realize that they understand. Their inquiry may have seemed simple, the re-creation with their particular rectangles of the mathematics done by the class as a whole, but it is in that activity and the conversation that drives it that the students have constructed the connections that they could not receive through two days of teacher-centred lessons. Mrs. Larkin has played a significant part in guiding the process especially with regard to the structure of the grid arithmetic element. Yet the students are more taken with their new awareness of how the variables connect the values for Rectangle A and the Original Square, and it is no coincidence that this is what their own words had been exploring and expressing. They are not getting their grid arithmetic done by following along with a teacher: they are inquiring into how it works and what it means.

In the next portion of transcript the students work to understand why the perimeter of their Rectangle A is the same as the perimeter of the Original Square and even Mrs. Larkin makes some new connections.

MsL: How would you do this? How would you get the perimeter of this thing?

Cl: Well you'd take,

Rb: *Thirteen times two, plus fifteen times two.*

Kl: *sos: minus one,*

MsL: Shouldn't we do the specific? That would help you.

Cl: So we go [with Robin in unison] *thirteen plus thirteen plus fifteen plus fifteen.*

MsL: Or.

Cl: *Thirteen times two plus fifteen times two.*

MsL: which one do you like better?

Cl: I like the times two one.

Kl: Is that right?

MsL: Is it? You tell me. Is that how you find perimeter?

Kl: Is that right, Robin?

Cl: Yeah.

Kl: I hope so.

Rb: Fifty-six.

MsL: Okay let's take it from specific to general. What is *thirteen* in relation to *side:*?

Rb: *side: minus one.*

Cl: So, *minus one. side: minus one.*

MsL: *side: minus one times*?

Cl: *side: plus one.*

Rb: No no no, it doesn't say *times squared*, it says, *side: minus one*,

Cl: *Times two.* Because if it's squared, that's *thirteen times thirteen*, we just want to do *thirteen times two*.

T2 Thank you. Excellent explanation, Colleen.

Cl: So *side: minus one times two*,

MsL: Plus, that's got to be in a big bracket. Right?

Cl: Hold on. Oh wait.

MsL: Because you have to do that first.

Kl: *Minus one* is in the bracket, xx,

Cl: *side: , minus one*, xx,

MsL: *Plus*

Kl: [with Colleen in unison] *side: plus one, times two..*

Rb: Great. You got it..

MsL: Now, did Mr. Mason go farther with that, or did he leave it like that yesterday?

Kl: It's kind of like this.

MsL: So we have *side: minus one times two*. What do we get here? *Two times side:* ?

Kl: *side: times two.*

MsL: Okay. *Minus two.*

Cl: Oh, all right.

MsL: All right, then let's do the other one. *Side colon plus one, times two.*

Right.

Kl: The twos cancel each other out.

Cl: Mm-hmm.

MsL: *Two times the side:, Two times the side:*, that's *four Sides*, right.

Cl: Yeah.

MsL: *Minus two*, the *plus two* is gone, that's all you're left with.

Kl: But that makes that the same answer.

MsL: That will be the same, answer as, if you just multiply the Side.

Cl: Yeah, cause it's perimeter, it's going to be the same. *fourteen times four* is the same as *thirteen plus thirteen plus fifteen plus fifteen.*

Rb: Oh! Right!

Kl: Okay.

MsL: Does that make sense?

Rb: Yeah. I was thinking Area.

Kl: It took you long enough.

MsL: I hadn't thought of it that way. But of course it would be the same. Of course of course of course. Because it makes sense how I got that.

Rb: Yeah.

MsL: Because this boils down to basically,

Kl: *side: times four.*

MsL: *Four times the side:* .

Cl: Is that us? Are we supposed to write that? No?

Rb: Let's do another one.

Cl: Which one is Piece B? thirteen by thirteen. Oh yeah.

Their exploration with Rectangle B proceeded as well as the exploration that has been carefully chronicled, but without the self-doubt or uncertainty of purpose. The Rectangle B in Appendix D is similar to theirs, a square whose dimensions are one less than that of the Original Square. Success with Rectangle B would be indicated by finding procedures for determining the area and the perimeter, using first numerical values and then *(side: - 1)* as its dimension. However the value of the success would be in understanding more fully the role of variable expressions for describing relationships, and in seeing the arithmetic relationship between two different but equivalent expressions for the same concept.

Such mathematical understanding is only one kind of understanding being constructed by Kelsey, Ruby, and Colleen in the inquiry and discourse shown above. They are also re-viewing prior mathematical understandings which they found relevant to their current inquiry. This includes Colleen's sense of integer operations and Kelsey's understandings of how perimeter formulas could be expressed in more than one way. The building of connections to prior concepts informs the builder about both the original and the new concept.

More significantly, the students are learning to learn in more complex and productive ways. Their growing enthusiasm for investigating the rectangle relationships as they continue is a direct consequence of their success in figuring out the many details associated with the processes. They are not as dependent on clearly delineated explanations and their memories as they had thought, and they are discovering themselves to be quite capable of learning in ways they did not know previously they could. Most importantly they are discovering and developing capabilities which are compatible with constructivist-structured learning opportunities. On they other hand they have not yet expressed that they have associated their success with their efforts to do this inquiry in ways compatible with such learning generally. It is too early to say that they have learned to inquire. This successful experience in the face of their initial frustration is only a starting point.

Students changing how they learn

> Trying harder may also be a strategy students will use, but, as mediated by a schoolwork module, this would amount to trying harder to do the task rather than trying harder to learn what the task is intended to teach (Bereiter, 1990, p. 616).

> Education [should be] conceived as a process of futuring, of releasing persons to become different, of provoking persons to repair lacks and to take action to create themselves (Greene, 1988, p. 22).

> Academic tasks vary in risk and ambiguity, and some students press teachers for assignments that are low in both factors. ... In too many cases, the teacher sets the performance goals, identifies relevant resources, establishes criteria for evaluation, and eventually announces winners and losers. Students generally gain recognition and approval by paying close attention to recommended procedures and taking few academic risks. ... We argue that if they are to profit from experience and meaningfully participate in their community, they need to learn how to cope with

stress, to become adaptive learners (McCaslin & Good, 1992, pp. 12-14).

Having students recognize the pedagogic purpose of an inquiry activity is of great importance. Kelsey made it clear that she was beginning the assignment with the goal of getting the assignment done. Colleen wanted to alleviate the anxiety she was feeling because of being so confused by the previous day's overview. Neither one saw their role in ways that would be particularly compatible with group inquiry activity. Students with Colleen's purpose, for instance, will be loathe to engage with confusion. Confusion can only add to one's concern about one's ability or about the appropriateness of the assignment. If a student assumes that confusion or ambiguity or an unclear sense of the process is symptomatic of something being wrong, then she is ill-disposed to succeed with inquiry-based instructional opportunities. The alternative to considering oneself inadequate for the task is to consider that either the task is inappropriate, or the instruction has been inadequate. Indeed Colleen's lack of confidence in the previous day's instruction was related to her initial orientation: she had written down everything she could as I provided an overview, and she and some classmates had insisted on further explanation whenever possible, but still the lesson had ended without the students understanding clearly how to perform these tasks. The next day she thought her task was to replicate those tasks by recalling the procedures from the day before. Colleen applied her traditional sense of a student's relationship with mathematics instruction to the lesson and the activity, despite the teacher's stating explicitly a significantly different intention.

Why did Kelsey consider task completion as their purpose? Perhaps she wanted to minimize her interaction with an activity which she anticipated would be difficult and whose purpose escaped her understanding. What might be the source in the misapprehension of purpose? During the teacher-centred lessons, the students participated in deriving formulas for areas and perimeters of particular rectangles, but clearly they are valuable as formulas only within the limited context in which they were developed. They have no practical value compared to area formulas they have derived before, such as for the area of a circle or the perimeter of a regular polygon. In other words, the value of deriving these formulas has to be perceived to be in the deriving, not in the outcome. Could Kelsey have perceived mathematics as the obtaining of answers, with the value of mathematics resting in the value of the answers? If so, she would have to be frustrated with a complicated journey whose destination is of clearly limited utility.

As I think back on instructing other students, I can remember helping students perceive the justification and/or derivation of various formulas: the area of a triangle compared to a parallelogram with identical base and height; the exponent laws from statements showing repeated multiplication; the surface area of a cylinder from two circles and a rolled-out rectangle; the tangent, sine, and cosine functions from similar triangles; the quadratic formula from completing the square. I taught all those formulas for their utility, but I taught their sources as connections for richer understanding of the formulas themselves and as particular examples of how generalizations of any kind are derived in mathematics. I remember that many students dutifully wrote the formulas down and waited through the lesson on its derivation to do the practice questions which consisted of applying the formula to calculate missing values. A side-trip whose purpose was to investigate the source of a formula was not of apparent value to students who perceived their role as students to be memorizing formulas and then using them to answer practice questions. It was only of value to students who wished to understand.

Colleen and Kelsey were not wrong. Given their experiences of what being a mathematics student meant, they were oriented toward behaving in particular ways because previously they succeeded from those orientations. Colleen along with everyone else was told at the beginning of the previous day that the lesson was only an overview and that they were not expected to understand everything on the first go-through. Kelsey, along with everyone else, was told that the task's purpose was to provide opportunities for figuring out relationships and expressing them with algebraic forms, not to learn and practice some new formulas. But a teacher's words do not determine a student's orientation to mathematics or the learning of mathematics! Just as with any element of mathematics, students need multiple experiences with a new concept before they understand, especially when that concept is a new personal orientation. They need to reflect on these new experiences comparing and connecting them to their prior understandings. Students will learn to improve their orientations as students just as they learn mathematics, by having new kinds of experiences which they make meaningful through reflective discourse and further use.

Colleen and Kelsey were having such an experience, and their comments which could be taken as purposeless venting could also be considered as reflective contrasts between what they are used to and what they were experiencing. It is clear from the beginning of the transcript that there was significant dissonance between the students' expectations of their context and the learning context itself. However, it can only be assumed by the students' success with the task and their growing satisfaction with it that they accepted the invitation which such dissonance represents to change their relationship with the dissonant element of the context, rather than demand that the context do the changing. The students would have preferred that their context adjust to their expectations for it, but somehow they made a choice to open their perspectives of the context to the possibility of change. They did not say explicitly, "Maybe there's something to this learning context that I'm not positioned well for, and I should explore it;" in fact, their early grumbling suggested little of the kind. Nevertheless they showed their capability and courage and openness by exploring in the face of the dissonance they had expressed.

Would it be worthwhile for us as educators to distinguish dissonance as a misalignment between a student and a classroom task and disequilibrium as a student's openness to being affected by the dissonance? So often in mathematics classrooms students step away from mathematics when they feel discomfort with it or with the instruction that provides their access to the discomforting challenge. Yet sometimes students benefit from their discomfort--indeed, students are not likely to change their orientations or behaviors unless there is some perceived awareness that their prior orientations or behaviors are not sufficient. How did the Rectangles Families task become a valuable and successful experience for these students rather than a frustrating event that would distance them from mathematics?

"Good tasks are easy neither to design and describe nor to replicate because they depend on specific students and on the classroom environment in which they evolve" (Reys and Long, 1995, p. 296). Rectangles Families as a good pedagogic task has the possibility of developing into activity which fosters a broader awareness of algebra. Its experiences can help students to perceive variables as standing for a range of values and to perceive variable expressions as being general forms for specific numerical instances. At least, that can be the outcome for students who are oriented to the complexity and self-directedness of inquiry learning. Without that orientation Colleen and Kelsey were unlikely to have succeeded without significant intervention, support, and guidance. Without that intervention and guidance Rectangles Families would have to be considered a poor task, at least for students with a relatively passive orientation to learning and expectations that learning mathematics should consist of clear and well-delineated tasks. Is it a matter of matching to students' orientations toward learning? If so then mathematics instruction cannot approach complex topics in complex ways, except with those few learners who come to possess an appropriate orientation. Clearly students must learn appropriate orientations appropriate to the processes which can lead to understanding mathematics.

For these three students, we cannot consider Rectangles Families as a curriculum event apart from the intensive teacher support which Mrs. Larkin provided. Similarly, for these three students the curriculum event must be thought of as including the students' openness both to express their frustration and to explore in ways they did not understand or like. Inclusive of these central interpersonal and personal elements, we could say that Rectangles Families provided Colleen and Kelsey with a learning experience which was valid mathematically but which also offered them a chance to perceive and to improve their orientation to learning mathematics. Maybe this is a quality of an exceptional pedagogic task. It is effective for those students who are aligned to succeed with it, and it helps those students who are not aligned appropriately to change their alignment.

This one example is still incomplete. Kelsey, Ruby, and Colleen have not expressed any enthusiasm for hard-earned mathematical understandings except within this one event where they had a teacher's constant support and guidance. They haven't shown that they would choose to engage in effective inquiry in the face of ambiguity without significant interpersonal support and guidance. However, they have shown how a curriculum event which is a mismatch with students' orientations to their roles can create growth rather than alienation. Because this kind of learning is so personal, the difference cannot be attributed to a curriculum design which is not inclusive of the encouragement and support for risk taking which is both context-specific and interpersonal. Just as this is true of mathematical learning viewed from a constructivist perspective, it is true of learning more about how one should and can learn mathematics. Others have acknowledged the centrality of realigning students' roles which acknowledge the interplay of the students' initial conceptions and the teacher's sense of more appropriate student activity by discussion of the process as negotiation or renegotiation: negotiation of culture (Brousseau, 1988); negotiation of role (Weiss, 1995); negotiation of "sociomathematical norms" (Yackel & Cobb 1996 p. 462), or renegotiating the "didactic contract" (Hoyles, 1988). All of these terms suggest the centrality of the students' senses of how math is to be learned to the success of mathematics instruction, and they all suggest the possibility of students and teachers achieving compatible orientations through ongoing interactions. Yet the terms' use of "negotiation" may imply that it is a matter of give and take between two roles within the learning context, rather than a deeply personal and ongoing process of changing one's relationship with the challenges of learning in different ways, as Kelsey and Ruby showed us in this chapter. We cannot underestimate the difficulty for students of abandoning familiar roles that have been successful to develop roles with uncertain rewards, if we are to identify ways to provide sufficient support and direction for the process whenever the students will encounter dissonance.

Independence and compliance

Before broader conclusions are possible regarding what the Rectangles Families activity meant in terms of learning to learn mathematics differently, a broader base of student experiences is necessary. Chapter 5 showed Silver, Monuel, Maria, and Hilde take the invitation which was the teacher-centred overview and engage independently with it. Kelsey, Colleen, and Ruby were much more reluctant, and ultimately engaged with the direct and constant guidance of Mrs. Larkin. The next group will be different again. From a role negotiation perspective, the intermittent involvement of a teacher (me) is a significant difference. However, the personal orientations of the students to themselves, each other, and to mathematics will provide a very different flavor to their engagement with Rectangles Families and to their learning to learn mathematics in new ways.

Like all the groups in the study, the personalities of Wai, Martin, and Ivan blended to create a unique team with its own approaches to learning through shared tasks. Martin considered himself the most conscientious of the three, both in terms of caring more about schoolwork in general and applying himself to achieve a high quality product for any assignment. Wai tried hard too, but was less willing to deal with details than Martin sometimes demanded, especially if extensive written self-expression was involved. Ivan was the most creative, but his attention often strayed, and he often distracted the other two as well. This group was less supportive of each other in their personal interactions than any of the groups portrayed so far, but they were all successful mathematics students.

Unlike Kelsey, Ruby, and Colleen, who had Mrs. Larkin's direct attention, these three students only have occasional access to a teacher's direction as they began to engage with Rectangles Families, because I am circulating among six groups. As the transcript below opens, Ivan sees me coming, and tries to make it sound like they are on track, although they aren't. Notice in their questions for me that they desire that the assignment be precisely defined for them in terms of what they are to write down. I dodge most of the questions, by getting playfully sarcastic with them about what should be worth marks, a game they eagerly join. They are much more comfortable with the openness of the assignment than Colleen, Ruby, and Kelsey were. However, they don't address the math until I walk away.

Iv: [Ivan] So we have to do the specific and general

Ma: [Martin] Kay, rectangles? I forget all this.

MrM: [Mr. Mason] Take out yesterday's work, and carry on.

Ma: What exactly do we have to do, though, Mr. Mason? We have to, kay, we have our rectangles, do we have to make, kay, each of them?

Iv: Huh?

Wa: [Wai] So we have to do specific and general,

Iv: Can I have a sheet of paper?

Ma: We only need one sheet to do this.

Iv: Oh really? Mr. Mason? One sheet per group, right, Mr. Mason?

Wa: We do it for our own.

Iv: It's one sheet per group, right?

MrM: Whatever works for you guys.

Iv: One sheet per group.

Wa: We all work on one together. We're not going to be docked for marks if we do this, huh? [giggles]

MrM: Oh, yeah, and neatness, and xx,

Wa: Well if we do really nice,

MrM: And whether you have a pen and pencil for today,

Wa: Uh, yeah.

Iv: Only a pencil.

MrM: Oh, that will be just half marks.

Wa: But my pen doesn't work very good.

Iv: I got a comb.

Ma: Are you kidding?

Iv: And a bus ticket!

MrM: Oh, five for you!

Wa: I got a calculator!

MrM: Oh, twenty! Yeah, you don't even have to work today, you xx.

Ma: I've got an egg!

Wa: Good! Good! [laughs]

Ma: Well I get extra.

I think transcripts like this one are seldom included in research reports. Yet it indicates so much about what the students thought of as important, and how they approached their work. Although it ended in jest, the students knew that talking about what is worth marks is how to find out in concrete terms that can guide their prioritizing what matters to the teacher. Although they knew that this work wasn't going to be handed in for marks, they and I knew that marks offered a vocabulary to prescribe priorities. I used marks in that role myself, when I suggested that getting multiple expressions for the same concept would earn a checkmark for each different one. Yet, for the sake of being more concrete and specific, talk about marks swings classroom conversations away from the actual learning goals toward synthetic goals about production, about what gets written down. That these two kinds of goals are not equivalent and that the distinction was not as clearly understood by this group as it should be will be made clear by this group soon.

In this next section, Martin, Ivan, and Wai scramble their way through the area of their Original Square (which for them is 13 by 13). They are very capable of disagreeing with each other, and do so over issues as varied as what counts as a square, which piece to do first, and who should write answers on behalf of the whole group. Martin contributes to the development of conversation by posing questions, both procedural and conceptual ones. Ivan has an alternate agenda: he wants to defend his right to misdirect the group's activity, but still be seen as a valued member of the group. His claim that a rectangle is a square devolves into a head-butting exercise with Wai that even temporarily includes Rob, a student from the next group who joins in just for fun. The contest concludes only when Wai accepts Ivan's invitation to stop being serious, and Martin pronounces what will be taken as correct, insisting that they move on.

Ma: Do we have to do it exact? Do we have to do it in general or individually.
Iv: First general no first individual then general. Rip this thing in half.
Ma: I am, xx
Iv: Well you already know the first rectangle, right, it's thirteen by thirteen for ours, right?
Wa: No, that's our square. Our rectangle is,
Iv: One side longer than the other.
Wa: A square is a rectangle.
Iv: No a rectangle is a square.
Wa: No! A square is a rectangle.
Iv: No a rectangle is a square.
Wa: Ask him! Ask Mr. Mason!
Ro: [Rob] A square is a rectangle.

Iv: No a rectangle is a square.

Ro: Yeah. A square is a rectangle. A rectangle is a square.

Wa: A rectangle is a square. A square is a rectangle, what am I saying? [laughs]

Iv: No a rectangle is a circle.

Ma: A square is a rectangle, for the last time. What did we call the sides, Wai?

Iv: *sos:*.

Wa: Side of Side. No. Oh, specific cases here, and,

Ma: Is this individual – this is general. . So what do we do?

Wa: This is Piece E.

Iv: No first we do,

Wa: Piece A.

Ma: Where's the one that is thirteen by thirteen, that's piece A isn't it?

Wa: No, that's our twelve by fourteen thanks to Ivan.

Ma: You guys are telling me it's different.

Iv: No no no! We do the square first!

Ma: Okay! Thirteen by thirteen. But what do we do here?

Wa: General. It's *sos:*,

Ma: *Side of Original Square, times Side of Original Square*.

Wa: I hope. What are we trying to find, anyways?

Ma: Area.

Wa: Oh, Area. Okay then yeah that's right.

Ma: Or, *sos: squared*, or *sos:* xx

Wa: Equals, Area.

Iv: Did you go skiing yesterday?

Ma: No did you?

Iv: Yeah.

Ma: Or, *sos: squared*. Or, *sos: squared*!

Wa: That's what I'm doing!

Ma: Well you got to put, let me do it, Wai!

Wa: I'll do it, I'll do it.

Iv: Martin, you're not the only one who got to write, and you're making it so ugly! [laughs]

Ma: Ivan's stupid, it's not our fault. He just does nothing.

Iv: What do you mean, I just do nothing. I'm the, brains.

Ma: Oh yeah, that's right, we have to put brackets.

Iv: I'm the one that told you *sos:*.

Ma: Okay, next square.

This group had much better days, but here we can see multiple examples of how they handled disagreements. To a degree this may have been a lack of skills in communication or incompatibility among the three, but in fact these students liked each other (although their style of interacting was admittedly quite feisty). Ivan did not realize or chose to ignore that Wai and Martin wanted to be more productive that day, and Martin at least was quite motivated to do this activity. However, I think there was a more fundamental epistemological factor determining their interaction processes: to resolve their differences the only strategy that the students used was to insist that they were right. The only alternate strategy was Wai's attempted appeal to an external authority, but with the teacher not fulfilling that role and without a textbook with answers in the back, the traditional external authorities for deciding right and wrong were absent. Did the three students believe that mathematical correctness or truth is dependent on the authority of the speaker more than the relationship of the claim to the context, or am I just interpreting as intellectual a passage that was just an interpersonal battle of egos or some community-building language play? The evidence from these minor arguments of what these students established correctness or truth is mixed. When Martin claimed that Ivan "does nothing," Ivan offered an example to justify his counterclaim. Yet all the other decisions which included some disagreement (what rectangle to do first, what label matches which rectangle, and what Wai should write down) were handled by insistence rather than by negotiation. There is no need to resolve this question on a single passage, but it must be regarded as one of considerable importance. If students in a constructivism-oriented activity space are to negotiate their understandings through interaction, surely it is of great significance how the students believe negotiation should be conducted.

Deciding what makes a good answer

> There is a certain point at which a person sees authority as an internal agent rather than as an external agent. At this point, truth is seen as eventuating from a personal perspective. It is here that one begins crossing the bridge from a submissive orientation to a position in which one's voice is a significant determiner of what one believes (Cooney, 1994, p. 628).
>
> Students will construct, but we want their constructions to be guided by mathematical purposes, not be the need to figure out what teachers want or where

> they are headed (Noddings, 1990, p. 16).
>
> Students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in these practices. These students can be contrasted with those who are intellectually heteronomous and who rely on the pronouncements of an authority to know how to act appropriately (Yackel & Cobb, 1996, p. 473).

The transcript of Wai, Martin, and Ivan's activity continues in the conversation from where it last left off. The students move on to Rectangle A without doing ant-trip for the Original Square. They quickly disagree about whether the order of the two dimensions matters, with Wai and Ivan suggesting that if in the specific or individual case they have already written *12 x 14*, then in the next case the smaller factor, *(sos: - 1)* must be first. The opportunity presents itself to judge the nature of the discourse, and whether it is through the use of personal authority or the use of evidence from context that the participants attempt to resolve the issue.

Wa: Next square is Piece A.
[Ivan is singing into the mike.]
Iv: Which is twelve by fourteen, which means you take one side, and add to another side.
Wa: Wasn't there another one? No there isn't, okay.
Ma: Area of A, equals *twelve by fourteen*.
Wa: Do we have to draw it?
Ma: No, I'm not drawing it. Why would we draw it. Twelve by fourteen.
Iv: Then you should number them.
Wa: 1, 2, 3
Ma: What did you say, Wai?
Iv: No, *take away one*.
Ma: No, *plus one* is fourteen, *take away one* is twelve.
Wa: Yeah, but you already picked twelve and fourteen, you got to do it right.
Ma: *Thirteen plus one* is fourteen, and *thirteen minus*,
Wa: Yeah but it doesn't equal twelve. You have to put it in the right way!
Ma: No you don't! That's just babies about. You don't have to put the right order, Wai, he doesn't.

Wa: Yeah he does.
Ma: No he doesn't, not in his examples he gave us.
Iv: Well, he's the boss!
Ma: I'm trying to be neat, okay?
Wa: It doesn't take that long to be neat, okay, it would be faster if you don't whine.
Iv: Martin, look. No, okay, that's right.
Ma: General's what?
Wa: *areaA:.* Because *sos: plus one, times, sos: plus minus one.*
Iv: Martin are you white?
Wa: He's Quebecois.

Ivan's question about racial identity is a tempting sidebar for me: although Wai could be ethnically categorized easily as being of Chinese descent, Ivan's heritage was not so clear, being east-European and Spanish American. The class was a mixture of many (and in quite a few cases, multiple) heritages, and although friendships and other interactions developed in the school without significant concern for ethnicity, many of the students had expressed concerns that in the high schools the students organized themselves primarily by ethnicity, meaning that many friendships which the students then shared would be harder to maintain, beginning with the next school year.

That Ivan! Whether it is skiing or ethnicity or whatever, he seemed to know exactly what topic was so inherently interesting to his audience that he could interpose it in front of the math and his audience will accept the bait. As the last paragraph shows, he even tempted me and I have interrupted the transcript just prior to a key exchange. Below, the group of students determines both $(13 - 1)(13 + 1)$ and $13^{2} - 1$ as answers but they make no attempt to compare them to each other or to a counting process with the actual rectangles. Instead, Ivan's purely instrumental approach (Skemp, 1987) to the assignment, to get as many different answers as possible, pushes them on. They replicate the two specific numerical answers in general algebraic form. Then, Ivan suggests that they could rewrite a previous answer, *14 x 12*, as *(14)(12)*, and Martin objects. Suddenly the purpose of their process becomes a relevant issue, but it fails to get resolved.

Iv: Isn't there like three ways to do this?
Ma: Well, you could go, *sos: , squared.*
Iv: Okay okay okay, you could go, *fourteen*

Ma: Ivan, just Don't!

Iv: Why?

Ma: Are there any different ways we could do it?

Iv: Go. Go go go. Go go go go go,

Wa: What's *fourteen times twelve*?

Iv: Go *thirteen squared, take away one*. That's another way.

Wa: It works right,

Iv: Yeah. That's free marks, again. We have to figure out as many ways. Squared, take away one. And then here's another here's another.

Wa: *sos: squared, minus one.*

Iv: See I thought of that. Remember.

Wa: Note: Ivan thought of it. Unquote!

Iv: And then go, *thirteen*, no no no, *fourteen, bracket,*

Ma: Around it?

Iv: *Twelve, bracket.*

Ma: Ivan, that is so, stupid, it's the same as the one up there!

Iv: [laughing] So what!

Wa: Yeah, but how do you put that into general?

Ma: Same thing as here but without that!

Iv: Yeah! [laughing]

Wa: No no.

Ma: We're wasting time!

Iv: Hey, that's four marks!

Wa: Note. Martin is finding

Iv: Are you crying, Martin?

Wa: And crying,

Ma: This is taking, look!

Wa: Ivan's making fun of Martin!

Ma: It's the exact same thing, Wai, why would we write it twice?

Iv: So why did Mr. Mason do it?

Ma: I don't know, but why would we write it twice it doesn't make sense.

Iv: Because we want to waste paper, and kill trees.

Ma: It's just a waste of time

Wa: Leave it now, leave it now.

Ma: Okay, but it doesn't make sense to.

Iv: Well, how many marks do you want, Martin? Three or four? Come on

Martin, answer.

Ma: Okay, Ivan, we could do *twenty subtract six, plus twenty, subtract eight.* And then we could do a *hundred, subtract eighty six,*

Iv: Good idea, good idea, let's go!

Ma: See, we could go on forever, Ivan, it takes too long.

Iv: Oh, crumb, Martin.

Ma: You want me to put *twenty subtract,*

Wa: No, don't Martin!

Ma: Okay, what's piece three?

Martin acceded about the multiple but trivial forms, and they avoided further quarreling about it by going on. In the moments that followed, they operated similarly with Rectangles B, C, and D, before I made another appearance. Below, we can see the possibility of a group reorienting itself in regard to its purpose without extensive guidance. I remind them about ant-trips, but instead of doing those together, we discuss the area of Rectangle E.

Wa: Is there another one we can do? Twelve by thirteen.

Ma: I'm lost. I think we're doing this all wrong.

Wa: You don't think we can go, *twelve*, and then *twelve plus one*, right? Mr. Mason. Martin says we're lost but I don't agree.

Ma: We're not lost.

MrM: Well if you're not lost, then you're not prepared to learn. So hopefully you're lost a little bit.

Ma: I am, a little bit.

MrM: Okay. He's changed his mind. So what you got going, guys? Which square you talking about?

Wa: Well we're doing one less, we're doing so many we don't, we're doing one less, you know, twelve by thirteen, ours is thirteen, so we took off one row from the top,

MrM: So it's twelve by thirteen. Okay, so one of your Anttrip answers will be *twelve plus twelve plus thirteen plus thirteen.*

Wa: Oh, we didn't even do perimeter!

Ma: So we have to do perimeter?

MrM: So if you want to do a general answer, you could do, *sos: minus 1 plus sos: minus one, plus sos: plus sos:.*

Wa: We just did the area.

Ma: We did area. Do we have to do perimeter too?

MrM: Yup. Well if it's too easy for you, don't do it.

Wa: It's too easy, yeah, it's too easy. [chuckles] Well yeah okay we'll do it because we can't really find anything for Area E, so we'll do perimeter.

MrM: What's wrong with Area E?

Ma: I left a space for,

Wa: Well it's just, *twelve by thirteen.*

MrM: You need to compare it to the original total square.

Wa: I'm not sure if that's,

MrM: Okay here's a thirteen by thirteen. I'm going to write Orig on it.

Ma: Okay.

MrM: This twelve by thirteen, [holds it in front of the 13 by 13 Original Square]

Iv: *156.*

Wa: Yeah.

MrM: Okay. Why is that?

Wa: Because, we took off a row.

Iv: One less column.

MrM: So, you could write down, *169 subtract thirteen.*

Ma: Could you really?

MrM: Or, *thirteen squared subtract thirteen.*

Wa: That's too simple, though. I thought you'd dock us for marks if we weren't more creative.

Ma: Or *sos: subtract Side:.* Or you could do, total. How do you say, total?

MrM: Well how did you get the 169?

Ma: No, what's the Total, how do you say it, like, in O S?

MrM: How'd you get it? How'd you get the 169?

Iv: You square it.

Ma: Oh so you could go, *sos:, squared,*

Iv: *Squared.*

Ma: *Subtract sos:.*

Wa: *sos:*, yes.

MrM: Does that make sense?

Ma: Yeah.

Wa: Yeah.

MrM: Now let's say I went over to Laurie's group over there where she's got nineteen by nineteen. And she took hers out that was the same in one direction and one row less. What would she do?

Iv: She would *subtract nineteen.*

Ma: She'd do the same thing we do.

MrM: From what?

Iv: *From, 361.*

MrM: *From 19 squared*, right.

Ma: She would use this, for hers.

MrM: What?

Ma: She would use this. She would go *nineteen squared, subtract nineteen.*

MrM: I see you're pointed at the general now.

Ma: Yeah.

Wa: Ah, yeah.

Although I cannot reconstruct what I was thinking at that time, I think that I considered myself to be helping them think about the relationship between specific answers and general ones. However, the effect which the students took from my intervention was that the way to find algebraic expressions was to find counting methods which relate directly to the rectangles compared to the Original Square. In the next section of transcript, after I start them off on an ant-trip and then leave, the discourse continues to use the relationship of answers to the actual objects as the determination of an idea's correctness and validity. They find it difficult to determine that the removal of one row of 13 squares would decrease the perimeter by 2 rather than 13 or 1, but because they can refer to the objects and show each other their thinking by demonstrating actions on the objects, they begin to pursue understanding through shared discourse. Ivan takes a back seat in this, but he does try to follow along, and it seems that everyone benefits when he asks for further explanation. Wai and Martin arrive at answers that make sense, and they express for most of their answers why those answers make sense. This is a different kind of justification than they were using earlier. Along with their growing understanding of the mathematical relationships, is their use of a different kind of justification evidence of a growing understanding about what would count as understanding?

MrM: xx you're just, you're making the method, and that's what it's all about. Give the Anttrip thing a try, and see if there's any good moments. You may as well do this one.

Wa: An Anttrip.

Iv: Perimeter.

MrM: Is the Anttrip for this one going to be bigger or smaller than the Anttrip for the original?

Ma: It would be smaller.

MrM: By how much?

Wa: One.

Ma: One square.

MrM: By one square? Or thirteen?

Wa: Yeah, *minus thirteen.*

MrM: Now, that's a good guess, and I'm glad you're guessing so quickly. That's how you make your learning moments happen is by guessing quickly. You're wrong,

Ma: I know.

MrM: And now you're going to think things through, because it's worth doing.

Ma: It's one less. It's one less on this side, see. Because if you take this one,

Wa: A column is one, you just take one off.

Iv: Okay okay wait wait. Here's a column.

Wa: No, a column less xx

Ma: Look, Ivan. If we fold this column over, then the ant walks one less. One less this way.

Wa: Can we just stomp the ant, and get it over with?

Ma: So it would go, *sos: plus sos: plus sos: plus sos: subtract thirteen*?

Wa: No, wouldn't it, it would go, *sos: and then sos: minus one,*

Ma: No okay look, We'd go, *sos:*, because this is right,

Wa: Yeah.

Ma: How much is this one though? 1 2 3 4 5

Wa: We're doing this, right? sos: minus one.

Ma: Ah, but Wai, watch now, and it would be *sos:, plus sos: subtract one,*

Iv: Wait wait wait. Look look look.

Ma: *Plus sos: subtract one*, because there's two sides that have one subtracted.

Wa: Yeah and then S, yeah and then S,

Ma: Could I just show you? This one is twelve, and this one down is twelve, but the one across is thirteen. It would be *sos: plus sos: plus sos: subtract two*.

Wa: What?

Ma: It would be *13 plus 13 plus 12 plus 12.*

Wa: Yeah but you said, *sos: plus sos: plus sos: minus two*. You didn't add the other *sos:*. So you must put, *sos: plus sos: plus sos: plus sos: minus two.*

Iv: I'm confused. [laughs]

Wa: Note: Ivan's confused!

Ma: Ivan, this one across is thirteen and so is this one.

Iv: No that's twelve.

Ma: No this one across is thirteen.

Wa: We're doing the twelve by thirteen.

Iv: Go the other way, so I can see the twelve.

Wa: Oh, please.

Iv: See how nice it looks now.

Ma: This is thirteen across.

Iv: No it's twelve. Okay it's thirteen.

Ma: Okay, so that's *sos: plus sos:* . And this is twelve, so that's *sos: subtract one*. So it's *sos: plus sos: plus sos: subtract one plus sos: subtract one.*

Wa: Cause we can't do *sos: minus two*, it wouldn't work that way.

Iv: *sos:* , *sos: subtract one*,

Wa: No wait wait, you could do that, you could go, *sos: plus sos: plus sos: plus sos: subtract two*. Wouldn't that work?

Ma: No because that's taking eleven.

Wa: Then it would be a trapezoid! [laughs] It wouldn't work.

This mid-stream interruption of a transcript is to question what the students meant when they decided if something would work. Although Wai's suggestion of adding four original sides and then subtracting two would get the right answer, and although it would be a different expression from anything they have yet written down, those are not all they now wanted of an expression. Instead, an expression works if it makes sense, if it matches a logical counting sequence. This changed how the students try to convince each other: they had objects with which they could show others their thinking. Ivan was interested enough in how an expression matched the objects it was describing that he asked for an explanation to be repeated! In the rest of this conversation, Martin and Wai attempt to capture the idea that overall the perimeter is two less for this rectangle than for the Original Square, but to portray it in a way that makes sense of the detail that the two that are removed are each taken from one of the sides, not all at once.

Iv: Explain it again, Martin, without using *sos:* .

Ma: Ahhh, Wai! But what if, Wai, we went *sos: plus, plus;* um *sos:, plus sos: times 2 subtract two.* Because if you t*imes 12 by two is twenty-four, subtract two is twenty-two.*

Wa: Why don't you just go *sos: times two,* and then *sos: minus one times two,* and then,

Iv: Stop right there! Okay, I don't like *sos:* . Let's change it to question mark.

Ma: No! No! Go ask Hilde if you can change it, and she'll say No.

. . .

Wa: If you go *sos: times two,*

Iv: Yeah.

Wa: And then you subtract two, that would only give you,

Ma: That's *thirteen times two,*

Wa: Yeah.

Ma: And then if you subtract two you get twelve, and then you add *sos:* ,

Wa: And then you add *sos: plus two,* that would work.

Ma: No no you would go *sos: times two, plus sos: times two subtract two.*

Wa: That would work.

Iv: That would work. . Okay, say it again.

Wa: Or you could go *sos: : times three and sos: minus two.*

Ma: That would work! Oh hang on. Couldn't we go, *sos: times four subtract two.* [Wai laughs] No really! *sos: times four subtract two*!

Wa: Yeah yeah. I was thinking,

Ma: There we go we have the answers. We've got a bunch of different ways to do it, Ivan! Whoo!

. . .

Ma: *Thirteen times four subtract two*, right Wai. *Thirteen times four subtract two*.

Wa: Yeah. Or *sos: times four subtract two.*

Ma: *Thirteen times two subtract thirteen times two subtract two.*

Wa: Let's explain the general now.

Ma: Aha! The fun begins, now, right, Ivan!

Iv: I hate this sos: sos:

[Martin in the background verbalizes as he writes.]

Wa: It takes too much eh. I think we wasted a lot of time on this. But it was fun though.

Iv: You know what I think about math?

Wa: What?

Iv: Time goes by so fast. And then in science the time goes so slow, L A so slow,

Wa: No kidding!

Wouldn't it be wonderful if Martin and Wai expanded on what made this process fun, or Ivan expressed what he hated about *sos:*? Unfortunately, although these students were making significant (but not instantaneous) progress at what counts as good learning, they do not readily discuss it at a meta-level. This is similar to the progress we saw Kelsey and Colleen making regarding what counted as appropriate learning processes for mathematics: they engaged in new processes, but did not yet express the differences or their significance to them. Ivan, Wai, and Martin were making progress at perceiving what counts as useful and meaningful answers, a decision which must be made (and which they are making here) repeatedly during inquiry learning. In other words, they are learning about what counts as learning (Keefe & Walberg, 1992; Wells, 1995). However, I must agree with Ivan's last claim: time does go by so fast, when involved in meaningful inquiry. The only thing I have found that makes time go by faster than small-group mathematics inquiry is trying to write a chapter about it!

Looking back

Both this chapter and the previous chapter revealed the diversity of students' orientations to learning when interacting a complex inquiry task. The chapters developed the significance of the students' orientations to learning and described how students can reshape their orientations as they engage in inquiry. These more complex themes about the students and their learning began with a particular inquiry activity, Rectangles Families, which ostensibly was a simple extension of the students' experience bases with algebraic descriptions of relationships among tangible objects.

As mathematics curriculum, what are Rectangles Families all about? At the simplest of levels, it is about a curriculum occasion which could provide a valuable follow-up for algebra tiles. In this sense, it should encourage teachers to let algebra tiles generally be with positive values only as you saw them used in Chapter 1, leaving for Rectangles Families the fostering of algebraic facility with positive and negative values. This could help students attend to the patterns in the arithmetic of algebra tiles, rather than attending primarily to the arithmetic itself. Also, Rectangles Families provides the students with the chance to perceive the variable as having value as a general label, to extend beyond the understanding that may emerge from using the algebra tiles. However, most important, it allows students to engage purposefully in the fundamental act of using algebraic expressions. Rectangles Families shows an occasion for recognizing the power of algebra for allowing conversation and conceptualization to shift from the specifics of a single value for a concept to a general expression for a concept, and back again. This may prove to be an experiential link with the algebra forms which emerge from inquiry into natural mathematical patterns such as Handshakes and the other Patterns inquiries. Rectangles Families can provide an occasion for recognizing the relationships among algebraic expressions as being subject to the rules of arithmetic. Concurrently, it can provide an occasion for students to practice and develop their inquiry skills, as we saw with the group which included Silver, Maria, Monuel, and Hilde. For an instructional sequence that might allow students to pursue these objectives, see Appendix F.

However, just as for the students, these Rectangles Families chapters could only view the mathematical and cognitive elements of this learning occasion in combination with other crucial elements. Our first hints that more was at stake than mathematical understanding was the variety of reactions to my insistence that the students generate the variable labels, and to adapt them as they found a reason to do so. This generated concern and conversation not only about the role of variables and their labels in expressing the mathematics of Rectangles Families but also about the role of the students in constructing an understanding of the activity and its content. I suggested that experiences and discourse about students' sense of themselves and of their role as learners/students be viewed as constructive learning activity of crucial relevance to mathematics reform.

I am proposing the reframing of discussion about the changing of students' roles as being a matter of *learning* rather than of *negotiating* roles or norms or contracts. This does not need to deny the mutuality, the interactive give-and-take which is highlighted by "negotiation," for these are significant elements of what we understand as teaching. Nor will the importance of role shifts or changes in norms be diminished by adding them to the learning goals of the classroom. In fact, identifying those outcomes as learning goals may make them more central: instead of claiming that they are fundamental to or essential to achieving the goals of a mathematics classroom, the shift states that they *are* part of those goals. For students to succeed in mathematics as the focus of mathematics itself changes over the grades, changes in the way that students learn must be a central aspect of what students learn. Such changes must be part of what teachers teach.

What is the nature of the changes which students must make in their behaviors, attitudes, and approaches to what they do as they progress into and through more complex mathematics? Such changes include the acceptance of new norms, the negotiation of classroom roles, the redrafting of a classroom contract. Yet Ruby, Colleen, and Kelsey showed that from the students' perspective, such changes are more personal than such statements imply. Wai, Ivan, and Martin showed that such changes are as much related to themselves as interactive people as they are to the particular context of the mathematics classroom. Attempts to understand and influence these changes must be taken from a frame of reference ready to be inclusive of students' senses of self, of mathematics, of knowledge and learning as well as their senses of school as an institution and the roles, norms, and contracts within the mathematics classroom. Although this broadens what we must view to understand the process, such a broadening can support a more efficacious understanding to the extent that the broader base of factors all interact with the target of our understanding.

How can such a broadening of our considerations include practical guidance for people concerned with studying or directly affecting students' learning to learn mathematics in new ways? Considering this as learning rather than negotiating allows constructivist theory about learning to inform what teachers can do to help students learn new ways to learn mathematics. As Chapter 3 suggests, learning to learn mathematics in new ways depends upon the provision of meaningful tasks which sponsor student activity inclusive of familiar and unfamiliar processes. The activity should include discourse about that activity, with students making connections among identified elements of the new processes and between those elements and the mathematical elements that concurrently develop within the activity. Students should also be guided to engage in reflective discourse providing them with the chance to notice what they did which was effective in developing the mathematical understanding and recognize themselves as successful and competent in learning in that way.

So far this chapter has provided a window onto only the activity within which students are learning mathematics in unfamiliar ways and the discourse which was within the activity itself. Although that discourse had yet to become reflective beyond the students' expressions of satisfaction with the outcomes of their activity, it portrayed richly the interactive support the students provided each other. Different kinds of interactive and supportive guidance from teachers were shown to encourage the students to engage personally in unfamiliar territory. Both groups of student learners will be the subjects of future chapters which portray more of their ongoing learning about what mathematics can be and how they can be successful as learners. The reflective aspect of learning in these realms will become more prominent, not only as they have more successful inquiry to reflect upon but as they learn to engage in such reflective talk.

With Ruby, Colleen, and Kelsey we were able to see capable students engaging in activity which stretched their sense of the role of students in learning mathematics. Inasmuch as this sense of the role of students is integral to their sense of themselves as particular individuals, I would claim that this is an element of their personal ontological learning. Both with them and with Ivan, Wai, and Martin, we were able to listen to students as they learned to judge what counted as valuable answers and what counted as understanding within their inquiry into the patterns and relationships of Rectangles Families. Because these factors are integral to what these students think knowledge and knowing are, it is appropriate to consider this learning to be epistemological learning. However justifying these terms and illuminating the nature of this learning cannot be completed within the confines of this chapter. With both these groups, although their discourse showed glimpses of their epistemological and ontological development, they did not talk about these elements in any terms which would support claims of fundamental changes. That is something which will form a central element in Chapters 6 and 7.

The next chapter provides a view of students and their changing perspectives over time. The changing perspectives on learning and mathematics of Martin, Wai, and Ivan will be portrayed through a different kind of data, their written interactions with me over the course of the research period.

Chapter 6. Discussing Marks and Developing Epistemologies

This chapter represents a shift in attention to a different theme. Until now, mathematics, the learning of mathematics, and the teaching of mathematics have been the dominant focus. Yet other elements have arisen which significantly influence and are influenced by the elements already discussed. The ways students approach the learning of mathematics and the ways by which they can learn more powerful and empowering approaches are the themes of this chapter. Metaphorically, the chapter suggests what early astronomers did, turning the telescope away from the moon, despite its apparent importance, and focusing farther into the night sky. Noticing the relative movements of planets and stars was an essential part of understanding the forces that affect the way the moon appears. Less metaphorically, the chapter suggests that educators focus their attention on significant elements which affect the learning of mathematics, rather than maintaining a singular focus on the subject area itself, to understand better the learning and teaching of mathematics.

During and after the strand of activity described in the last chapter, the balance in pedagogic intentions made a significant shift toward the elements that were affecting the students' learning of mathematics during the interactive small-group inquiry instruction. The inquiry instruction itself never stopped being about math--in fact, the attention of the students remained singularly focused on their algebraic understanding. However, the ways in which the students learned about algebra was complicated by the students recognizing (and being assisted to understand more fully) factors which affected their mathematical learning. As Chapter 5 suggested, students' senses of what counts as mathematics, their senses of themselves as students, and their senses of what marks and school are about serve as limiting factors for how well students can learn mathematics, unless their understandings of these elements grow along with the mathematical understandings. This is especially true when students are guided to learn mathematics in ways different from their previous experiences and their expectations, as was the case in this research.

This chapter claims and provides evidence that students' epistemology is a necessary consideration for mathematics teachers embarking on instruction organized around a constructivist perspective. It uses the term *epistemology* to point to the students' understandings of what counts as understanding. This use includes what counts as mathematics, and what counts as knowledge. Such a use of the word is not traditional (Duran, 1991; Greene, 1994; Orton, 1995; Sorri & Gill, 1989). The chapter shows that each student's understanding of understanding could well be unique (Bateson, 1979; Dewey, 1960; Rorty, 1991), no differently than each student's understanding of algebra is likely to be a unique combination of personal and interpersonal experiences and meanings. Also, the chapter demonstrates that the students' understanding of understanding will not come first from considerations of generalized claims about understanding but by consideration of the one example of understanding which they have the opportunity to notice and affect: their own. Finally, each student's epistemology is shown to be dynamic (Dewey, 1916/1966; Cobb, 1995c; Varela, Thompson & Rosch, 1993), subject to change through further learning about understanding.

That means that this chapter will make two claims. One, students' understandings of what counts as understanding, both generally and for mathematics particularly, matters significantly to their learning of mathematics. Two, students can learn to understand more richly what counts as understanding. A corollary of this latter claim is that students' epistemology can be the subject of teaching, suggesting that the constructivist conceptualization described in Chapter 3, that students can come to understand through small-group cooperative inquiry with multiple layers of reflective discourse, is applicable to teaching epistemology much as it is applicable to teaching algebra.

Developing these claims requires looking at particular students over a longer period of time than in previous chapters. To that end, this chapter will feature Martin and his group-mates, Ivan and Wai, who were featured in the latter half of the previous chapter. The chapter will show them dealing with epistemological issues as they arise within their ongoing learning of mathematics, as they notice aspects of their own learning. In order to show epistemological differences not only among the three students but for each of them over the timeframe of the study, the chapter makes extensive use of the students' own written words as much as possible.

The homework

> "Real talk" reaches deep into the experiences of each participant; it also draws on the analytical abilities of each. Conversation, as constructivists describe it, includes discourse and exploration, talking and listening, questions, argument, speculation, and sharing. ... In "real talk" domination is absent, reciprocity and cooperation are prominent (Belenky, Clinchy, Goldberger, & Tarule, 1986, p. 144, 146).

> My proposal is that social scientific practice based on a commitment to knowing people as well as possible is a worthy epistemological paradigm tout court (Code, 1991, p. 41).

> Understanding how people make meaning of their experience stems from listening to what they have to say about it. (Perry, 1970). Listening is difficult work (Baxter Magolda 1992, p. 1).

> In feminist methodology, the researcher recognizes that the better the research participant is known and can become involved in the study, the more likely it is that the researcher can come to understand the real issues and views of the participant (Fennema & Hart, 1994, p. 657).

"Homework" was the label selected by me as teacher and used by members of the classroom to refer to the students' responses to a reflective question I asked at the end of each research day. (For a list of these questions and four examples of their original appearance, see Appendix G.) The students could write as much or as little as they wished, including handing in only a piece of paper with nothing except their name and date. They were clearly told that this homework would not be for marks, but that I would read whatever they wrote and that I would write a personal reply of at least equal length.

I chose to call this process "homework" for two reasons. One was to create some dissonance over time, through a comparison with the homework that was such a key part of traditional teacher-student relationships. The second was more immediate, however: with the purpose of the process so unclear to students, I hoped that the label itself, despite my assertions that it wasn't for marks in any way or for practice or for developing work habits, might encourage students to engage. Some students did suggest later that they had indeed engaged with the process because they knew that anything called homework wasn't to be trifled with, but as I came to know these students better, I think that they would have given the process an honest effort just in response to my clear desire that they do so, without the manipulative label.

What was the purpose of the homework? Here is the answer Martin gave to that very question. It is followed by my reply in italics. Even though my replies are not central to the chapter's theme, I will provide my replies for all the homework data used here to reflect the dialogic integrity of the process.

Martin, January 27--*Homework*: *What have the homework questions been designed to help you with?*

Homework is used for you to find out about us as well as vice versa. As well it lets you see how well we are doing (in some cases). It lets you see how we're thinking as well as our opinions and what we think.

The homework questions have helped me in many ways. First I believe that some of your questions are designed to help me think about what is coming next. These questions give me a chance to think about topic related questions that will be coming up next. Some of the question are designed so that you can find out what we are having problems with, and you can make changes or give extra help. Some questions are related towards my opinion on various topics. I believe that maybe you use my opinions and the way I think in your research perhaps. Again my opinion (as well as others) may help you make changes in our work and how to explain it.

I am happy that you believe I listened to your words, Martin. You suggest I used them in my research, and to make changes in what I taught and how I taught. You suggest that I gave you extra help when your homework suggested it. To be able to please you like that would make me proud of my teaching!

Your first sentences suggest that the design of the questions was so that answering would help you. Your suggestion that sometimes they helped you prepare mentally for a topic to come is correct -- I'm glad you noticed!

I will only be here twice a week at most from now on. If I don't ask a question, feel free to ask one yourself, and I will respond.

As Martin suggested, the homework flow was ultimately data for research, as I had freely stated when students asked. However, I believe that I never asked a question because I perceived a particular need for data on a topic. Rather, the questions were chosen to support students' ongoing struggles with the learning challenges of the research. I believe that the questions often sponsored thoughtfulness about the learning that was going on, and the responses I provided were with the singular intention of supporting the student in the risks they were taking to learn in more complex ways. Certainly their feedback helped me to shape my instructional choices as well. Thus, the homework questions I asked and the replies I gave to each response were pedagogic in intention and effect (Wilson, 1995).

Here are Wai and Ivan's responses to the same question. Wai was thoughtful, persistent, and conscientious, but was not nearly as verbal as Martin. He expresses that the homework questions were for the sake of the students' learning, and demonstrates the format's value to both of us as a way for a student to reverse the normal direction of questions and ask me what mattered to them. The test and marking process which he mentions is discussed further in Chapter 7. For now, Wai's question may be considered an example of what he describes as being the homework's purpose to him. I would rather than he elaborated on his use of "learning experience," a phrase that sounds too much like my talk to be taken at face value even with a thoughtful person like Wai.

> Wai, January 27--*Homework: What have the homework questions been designed to help you with?*
>
> The homework questions are designed to help us to understand the concepts we learned the previous day. It also improves our learning experience. Oh, by the way, why didn't you give us an exact mark on our test?
>
> *Your answer mentions two key ideas. The first is the math concepts you learned, and I think I can see how the homework questions have helped you there. The second is about improving on your "learning experiences". This is a more nebulous, but more powerful, idea. How did homework help, I wonder?*
>
> *To give you an exact mark, I would have to decide what exactly each question was worth and what exactly a right answer should look like. To do that I would have to ask questions that have specific right answers. Also, I would have to ask everyone the same questions, so I could compare.*
>
> *Instead, I asked open questions that gave you guys space to grow. There was room for you to control your own learning processes. As well, you got to choose, not only what kind of test, but during the test you continually had options about how useful your answers would be to the group. As a result, I could notice and describe some of the complex ways you learn, but I could only give you "an exact mark" if I made one up.*
>
> *What exact mark would you give yourself, Ivan, and Martin, Wai?*
>
> *Thanks for the tough question!*

Ivan's replies were always brief. He seldom gave details and may not have thought deeply about the questions I posed. When he answered, his answers were always polite but general. I always had a feeling that when Ivan answered he was taking his best guess at what he thought I wished him to write. He was not being disingenuous, but was rather being true to what he thought was how homework was supposed to be done. Replying to Ivan's responses was always a dilemma for me. I wanted to push him into more thoughtfulness but I didn't want to be pushy. Could I somehow suggest that more thoughtfulness would be a good idea while acknowledging the current value I could find within his limited replies? With the luxury of a retrospective stance, I would say now that my reply on this occasion fails to be gentle in hinting that Ivan could respond more purposefully in future.

> Ivan, January 27--*Homework: What have the homework questions been designed to help you with?*
>
> The homework questions help you, Mr. Mason, understand what I'm doing and thinking. It helps me by understanding my homework and what you want.
>
> *"Understanding" seems to be the key idea in your answer, Ivan. You mention mine, and yours. To be honest, I still am not sure of "what you want". Maybe more homework would help!*

I believe that the homework questions represented a dilemma for Ivan, even more than replying to him represented one to me. Ivan knew why teachers asked questions: they wanted to catch students who didn't know what they were supposed to. But why then did Mr. Mason ask these questions that didn't seem to have right answers, and why did his two colleagues seem to appreciate the questions and answers and replies so much? Ivan knew why teachers gave homework, too: they wanted to develop students' industriousness and grade them for it. But why then did Mr. Mason keep insisting that students could choose how much or how little to write, and why did he make it so clear that this homework wasn't for marks or for any other system of reward? On occasion, Ivan asked these questions of his two partners, but his intent was more rhetorical than his partners assumed: he was stating frustration with issues of dissonance more than he was expressing a desire to understand. I believe that exploring what the homework could mean to him would have threatened his sense of how teachers and students related, and this was something he wasn't ready to do. In this sample, Ivan recognized and expressed that I wanted to understand learning and that the homework was part of that, and he recognized that the homework was supposed to help him understand something, too. Yet he didn't express the role of homework to help him understand, except with the mathematics and tasks in the class. He is not consciously epistemological in his response.

For most students the homework developed a particular purposefulness, and that purposefulness was mutual. Overall the homework began as a written exchange of information with whatever each of us wrote being intended to inform the other person. It soon grew, however, into more than that. It was a way to influence the other person, to have an effect on someone of significance. It was a way to think more about something and have someone else involved in that process. It became a safety net for students, a way for them to elicit my support (intellectual or affective) as they dealt with variations in confidence. Similarly it was a safety net for me, one that was especially valuable on occasions that I encouraged students to pursue challenges that might be beyond their comfort zones. It also became a way for a student or for me to provoke the considerations of the other. In other words, it was a communication channel and a vital element of the pedagogic relationships in the classroom. Only now does it serve as a record of students' ongoing efforts and thinking.

Students' awareness of learning processes

The homework samples above make it apparent that students can discuss learning activities in which they take part and can ascribe purposes to them. Yet it is more difficult for them to discuss learning in its abstract sense when a particular context and activity aren't specified. Is this a deficit of the learners, a characteristic of adolescents, or is it attributable to the subject of the description? Could learning be a process that resists description? Imagine that Auguste Rodin has been asked to follow up his masterpiece, *The Thinker*, with a piece to be called, *The Learner*. Rodin wants to catch the act of learning in mid-process, and by borrowing a contraption from his good friend H. G. Wells he has crossed space-time to asked for our help. We are to mentally sketch a figure of a person in the act of learning. Is the person reading, or writing, or talking, or listening, or doing something manually, or doing the odd-numbered questions from page 263? Learning is a hard verb to represent. In fact, "to learn" is a verb that does not signify a particular action. It is not a reflexive verb, although the ontological statement, "To be is to learn" is pleasing to me. It is simply an action verb which characterizes an action by a particular outcome rather than its process. If we intend to guide students as they learn about learning, it will be important to understand how they perceive it and how they describe it. In this process Martin, Ivan, and Wai can help.

Although they do not refer to it as learning, students certainly can describe the actions they do. These first examples which follow are for me a celebration of the richness of the activity involved in the inquiry approach to algebra. On a day relatively early in the research, I had announced that during the next period we would be beginning the next strand of activity, called The Painted Cube. However, the students put the brakes on with a screech. We hadn't finished the three previous activities! My protests, that those activities could never be finished totally because each part which one comes to understand shows parts which could be understood further, fell on deaf ears. We hadn't finished, because I hadn't provided closure. The students wanted to hear from me what the answers were, even though they knew already that they knew them. Of course, they were right that closure is important, and I was wrong to think that closure could not occur when learning is natural enough not to have artificial endpoints. We did indeed have a rich conversation about the salient details of Handshakes, Odd Jobs, and Passways. At the end of the period, however, I had not pointed them toward the next topic, and I did what teachers have done many times before with unmet objectives: I assigned it as homework! We begin with Martin's two-part answer, which is to my ears a rich list of the "ing-words" of actively learning algebra. Martin's answer will help to form a sense of Martin's view of mathematics and how it is to be learned.

> Martin, October 29--*Homework: What do you think is coming up with the Painted Cube?*
>
> I think painted cubes are leading up to model making and understanding how to get patterns. I think it will also lead to other similar problems. I believe we will be using algebra and making charts (increments). We will be finding patterns using the number of core, number of sides painted on corners, center faces and edges!
>
> I forgot my quality one at home so I did this one at lunch today. I will bring my quality one in next time you leave.
>
> *I will answer your quality one, Martin. You can have this one back for now.*
>
> *What do you think is coming up with the Painted Cube?*
>
> I think it will lead up to new problems related to painted cubes. (Ex. first we did Handshakes then Odd Jobs and then Passways.) I think that also painted cubes will lead to us (groups and class) finding solutions using the number of core, corners, edges, and the number of center faces painted. I believe we will be using algebra and making charts to show relationships. These are the things that I think are coming up in the next few classes. I think we will also be using the models we made to help us get solutions as well as ways to get and solve the problems. We

will be using the models as visual aids.

The painted cubes project is 6 questions (at least) in one, Martin, so we won't need to add more for a while. You mentioned some of the them in your answers. The charts will help us see patterns and make graphs. The algebra will describe the relationships.

I am pleased that you mention how important visual aids are. One of the reasons I used them in today's class was because of your previous reminders.

Martin shows the importance to him of the multi-faceted elements of the process. With the patterns contexts they had been exploring, I had thought of the charts (e.g., number of people in the top row, from one to twenty; number of handshakes in the bottom row, from zero to ???) as a simple record of answers to make graphing more straightforward. However, in the charts Martin (and others) had seen for the first time the power of multiple answers to a relation arranged in a sequence, and it was in the charts that the increments between answers had become visible as well, rather than in the graphs as I had expected. I think Martin used *algebra* as his label for the arithmetic statements of relationship among variables, and models and visuals were sketches such as the one which makes the Passways question the same as the question to count the diagonals in a polygon. Although I do not think Rodin would know how to cast it in bronze, Martin held the pattern-finding aspect of each inquiry as being the centre of the learning process, and recognized that comparing the patterns found in different contexts was part of the challenge. I am excited even now to read that Martin recognized so early in the inquiry instruction that his task was not just to learn the answers to the patterns but to learn the methods to learn the answers, to "get solutions as well as ways to get and solve the problems."

His partners' answers are not uninteresting. Ivan chose to write no homework before class, but was not willing to hand in nothing after seeing his partners' responses. He scribbled down what he could as the period started, and as he handed it in said, "I hope this is right." For me, as I wonder whether to include this in this flow, it is more than just the challenge of maintaining the unity of the group: in fact, Ivan's answer reminds me now that epistemology and algebra and research are always about the students' own perception of what mattered to them, and what their role as student was about. I now believe that the scolding within my answer was more likely to reinforce Ivan's sense that the teacher's standards were what mattered than to provoke a deeper approach next time, but perhaps by scolding I was simply adopting a stance Ivan found comfortable in its familiarity, despite my intentions not to revert to Ivan's sense of archetype.

> Ivan, October 29--*Homework: What do you think is coming up with the Painted Cubes?*
>
> I think it is leading up to algebra. Understanding of area and patterns.
>
> *If you took the time before class to answer, you could give a longer answer, Ivan. Your answer is right, but not specific.*

Wai's homework that day also reflects in its language the same sense of student-centred activity which was apparent in Martin's. Although I did not appreciate it at the time, it also captures a different aspect of what algebra is about than Martin's does.

> Wai, October 29--*Homework: What do you think is coming up with the Painted Cubes?*
>
> I think that the painted cube will lead up to us doing algebra with patterns. That means we will be able to do and figure out patterns without doing specific cases. We can figure out the pattern immediately.
>
> *You are right, Wai. Algebra is a way to discuss patterns without specific cases. Being able to discuss patterns is difficult, and algebra gives us a hand with that.*

I did not at any time discuss with the students what "algebra" was. It was the word we used as a label for what we were doing. Its function was as a label for what we did: algebra was Handshakes, Passways, and Odd Jobs, the class and group activity, the record sheets and graphs and sketches, and this homework. As Wai's understandings continued to develop before me, I perceived the hidden depth in this quick description of algebra first as patterns and then as generalizations of patterns that incorporate specific cases. In the correction which Wai offered to the standard definition of algebra as patterns, figuring out those patterns was central. In any case, Wai found a pattern of his own, one which unites the three specific inquiries he had experienced. Perhaps all learning (at least the learning in inquiry contexts) requires us "to do and figure out patterns." "By using mathematics to organize and systematize our ideas about patterns, we have discovered a great secret: nature's patterns are not just there to be admired, they are vital clues to the rules that govern natural processes" (Stewart, 1995, p. 1).

Students' awareness of learning itself

> Defining specific strategies for each way of knowing tends to promote rigidity, rather than flexibility, in practice. Flexibility is essential given the dynamic nature of students' development, the diversity of ways of knowing and gender-related patterns in any given group of students, and the variety of sources for confirmation and contradiction. ... Thus, I propose that we approach education as contextual knowers ourselves -- that is to say, we should recognize the importance of context as we make decisions about educational practice (Baxter Magolda 1992, p. 225).

> Learning to simplify is to climb on your own shoulders to be able to look down at what you have just done -- and then to represent it to yourself (Bruner, 1979, p. 101).

> Learning, and learning how to learn, give us freedom from oppression. Meaning, and controlling meaning, is the key to oppression (Gowin, in Greene, 1988, p. x).

Despite that students tend not to use the verb "learn" without prompting, it would be inaccurate to say that students cannot discuss learning. Rather, they choose to make sense of what they do in other terms, often choosing to do so even when I phrased the homework triggers to discuss learning directly (Appendix G). Within the homework responses provided earlier in this chapter, students were able to talk about purposeful complex activity such as finding a pattern by making a chart. Talking about it is not straightforward, but the students clearly describe the activity as a goal directed task. It is significant that the to learn is not the expressed goal. Rather, it is to complete a chart and find a pattern and find a way to describe the patterns. The students are aware that the charts and patterns as products have no use to them; in a sense, they serve as ends-in-view for means which they value. They discuss their learning by discussing the producing of the ends-in-view.

This focus on producing products was a complication for me: I had a strong desire for the students to shape their learning processes toward richer activity, and (as the above responses suggest) I was feeling quite successful. Yet, there was a gap between the levels on which they and I tended to discuss cognition. *Learn*, a significant element of my personal working vocabulary, wasn't particularly useful to the students for their own processes of describing what they were doing and why they were doing it. They had no difficulty in dealing with my use of the word, because when I used it to ask what they were doing or thinking, they would describe the set task (such as completing a chart) and the sense that they made of it (such as the constant pattern in the increments between values). They could talk about the learning they were doing and they could describe what was significant to them about it, without characterizing it by using that particular word.

In fact, this points to a distinction between my understandings and that of my discourse community, where a taken-as-shared understanding (Cobb, 1995b) may not have been shared at all. I now have the opportunity to re-view the blank stares I have so often received when encouraging a student (in many contexts, not just this research strand). "Now you're learning!" I would exclaim. My intention was to celebrate a particular act, and perhaps to point them toward noticing a particularly effective behavior. I think now that behind those students' blank eyes were the words, "Learning? All we were doing was what we were told to do." Their intention was to fulfill their role as the teacher delineated it by specified performance tasks, and even when they aggressively pursued sense-making during the task, ambiguous verbs like *learn* didn't specify any particular behavior.

There was one condition that did make it easier for the students to talk about learning as learning. "Learning" seems to serve us best as a retrospective verb. It is as if it is easiest to attribute this outcome-dependent label to actions when we are looking back on them. Such usage seems to suggest that "learning" would be the perfect verb for metacognitive reflection, but because the subsidiary outcome (whatever has been learned) must already have come into view, it is not particularly amenable for reflection on recent action. I found this out about two weeks after the homework exchanges shared above, at the end of a day just prior to the day report cards were to be sent out.

It had been a particularly positive and valuable class, and at the end of the afternoon I asked the students to describe a moment from class which showed they had learned. The students reacted to the request as if it was absolutely impossible. One student's comment was, "Learned? We haven't learned! We've been working the whole time." Another student said, "But you didn't teach us anything. You just wandered around from group to group!" These statements make apparent the limited sense of learning which they were used to using in school, that learning is about listening to teachers, and talk about learning should begin with the teacher and be about what the teacher has specified as content. However, it was also complicated by my request that they reflect (in terms of learning) on very recent events which had not yet been completed. We renegotiated the question to allow students to talk more safely about learning that had been concluded. The first answer is Wai's.

> Wai, November 19--*Homework: Describe a learning moment from a different class today.*
>
> Another learning experience I had in another class was in science where I learned that some things appeared to be different but it turns out to be the same. What I meant was that I learned that volume and mass have no effect on density if the material you are measuring is the same. So this means something's appearance is not the same as a material's real property. This can be used in mathematics like how some formulas look different with different numbers but are the same.
>
> *Nice example, Wai. In math together I have heard the formula for the area of a rectangle as*
>
> *-- l times w*
> *--length times width*
> *--base times altitude*
> *-- base times height*
> *-- number of squares in one row times number of rows.*
>
> *These are different formulas but only one relationship.*
>
> *Density = Mass / Volume*
> *DV = M*
> *V= M/D* *Three ideas, one relationship, a zillion formulas.*
>
> *Why does everyone always try to memorize all the formulas?*

The common understanding among the students about what learning meant was to be told something by a person who knew it, to remember it, and to tell it back to the person. Here we see a much richer sense. Wai is saying in straightforward everyday language of his own what he learned about substances and their density. Is he only reciting words he formulated while developing this understanding earlier? That is unlikely--Wai was not a talkative student, and science class by their accounts was not an environment that sponsored student self-expression. Instead it is likely that Wai was putting his understanding of density as a property into precise words for the first time, and was achieving clarity in his understanding by doing so. Wai was consciously transforming his new scientific sense into the language of common sense and he was clarifying for himself his own understanding in the process. Instead of debating an interpretive claim, the reader could judge the last sentence of his response. Wai's last sentence most clearly captures the sense-making that complements self-expression. Here was something forming in his mind just as he wrote it -- a comparison between a substance's density staying the same despite how much of it there is and a formula's basic meaning staying the same even when the formula is rearranged or recoded.

For Wai, describing a learning moment was itself a learning moment. Wai himself noticed this happening two months later, after having explained something from his test in front of the class. "The presentation let me explain my own test and let me improve on some of my own answer because when I spoke them out loud, some of my answers did not make sense, so I changed them to a more understandable answer" (January 26, homework).

Martin's answer is more purely descriptive. He provides us with a nice example of the role of actually using a concept to understand it. In turn, my answer was an attempt to suggest that any understanding is likely only a partial one, but one which can support engagement with further questions that are not yet straightforward. I knew that answers like this needed to be used sparingly, and that I couldn't expect any one of them to have significant impact.

> Martin, November 19--*Homework: Describe a learning moment from a different class today.*
>
> The learning experience that I had was in social class. We were doing longitude and latitude and much of the class was confused. So the teacher gave each of us an atlas. We turned to different pages and she gave us coordinates for different cities. After a whole lot of examples the class had the ability to use longitude and latitude.
>
> *I like longitude the best. At the equator, how could you figure out the distance between 1 degree of longitude? At the North Pole, what would it be? What about at the 49th parallel? How wide is a time zone?*
>
> *I think these questions come after you can use longitude and latitude. Are they math questions, or social studies?*

Ivan's answer also provides a window on the complexity of teaching and learning in classrooms. It seems as if Ivan heard a different question. He gives a description of how learning at this moment is different from last year. Ivan not only puts "understanding" as his central purpose in this kind of mathematics, he contrasts it with a seemingly purposeless approach the previous year. I'm pleased now to be sharing a time when I gave Ivan the kind of supportive and encouraging answer I feel should have been my regular choice, considering the personal changes he was facing and the risks he felt he was incurring. With this more positive example, I will stop apologizing for how my replies appear to me now.

> Ivan, November 19--*Homework: Describe a learning moment from a different class today.*
>
> A different learning moment from a class today is when I was in Grade 8. I was just taught and given homework to do for the class. Getting the answer was more important than understanding the work and the formula. Also we weren't allowed to speak our thoughts.
>
> *I agree that grade eight is usually done differently, Ivan. Often, grade nine is done the same way. Very few people can get all the way through grade 12 that way, though. Also, although it is a more straightforward way, it is a less interesting way to learn. When you are interested in how something works, it is easier to understand it.*
>
> *Thanks for the answer.*

This series illustrates a distinction between having students learn about learning and having students learn about their learning. As shown within this sequence of homework responses, the students are able to learn, and they are able to reflect upon their learning, but their learning isn't about learning in the abstract or as a general phenomenon--the object of their reflection is their pursuit of mathematical understanding. Their understanding of understanding is developing, but it is not the subject of direct instruction or the purpose of the original activity. Thus the intellectual pursuit of mathematical understanding became an experience for the students to consider for this second purpose. In reflecting upon that pursuit, they could learn about learning by drawing meanings from their own experiences, just as they had learned mathematics by drawing mathematical meanings from those experiences. The learning about learning was within the context of their learning mathematics through inquiry and in its turn it enriched the mathematical inquiry which followed. Although the ability to learn in more complex ways can be a significant goal in itself, in the case of this study that ability was pursued as a necessary means for students to succeed more completely with their mathematics.

The currency of marks

> Knowing in more complex forms was possible when students' genuine relationships with authority replaced detached or apprentice relationships (Baxter Magolda 1992, p. 223).

> When curriculum includes only the plans teachers make to deliver instruction, the child who emerges is usually what we might call a "school child," one who is either compliant with or defiant of the exercise of institutional power (Clifford & Friesen, 1994, p. 7).

> Commonly, mathematics is associated with certainty, with knowing, with being able to get the right answer quickly.... These cultural assumptions are shaped by school experience, in which doing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher. These beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing (Lampert, 1991, p. 124).

Generally, the students in this class were used to receiving good marks. They believed themselves capable of success and were motivated to succeed. Good marks were of significant importance to them. For them, marks were the language by which their success could be described. Perhaps there were longer-term agendas toward which the students aimed their efforts, but marks provided a vocabulary for setting short-term targets and they provided the teacher with a vocabulary for specifying what mattered most to them at any given time. Marks provided a discourse space for what mattered now. This currency in relation to time intermingles teasingly with the image of marks as *coin of the realm*. In the eyes of these students, marks were earned by their efforts and their behavior, and marks were the payment they received for the products (homework, assignments, presentations, tests) they created. For instance, products that involved more work were supposed to be worth more marks.

What are marks? To the students, marks were not a hard thing to specify. There were *marks*, a percent score attached by a teacher to a student's product or a series of student products, and *effort marks*, a categorical description of much less significance included at report card time. To many of the students, marks were a relatively unambiguous structure. The classic interpretation of marks seemed to be based on tests with many separate questions. If a teacher asked 25 questions on a test similar to questions the students had done for practice or similar in content to material covered in class, and a student got 20 questions right and 5 questions wrong, then she/he received 80%. Yet, even in traditional examinations, this was not so straightforward. Somehow, the test had to be designed to be not too difficult and not too easy, so that the student who deserved 80% got 80%. It was this underlying meaning of marks, that marks identified the achievement each student deserved, that made the percent grade meaningful for essays and group projects and any other student performance or product: a student could get 4 out of 5, or 80%, without anyone questioning, "Out of 5 *what*, out of 100 *what?*" As with any medium of exchange, the meaning of marks was determined by their usage.

Martin was not critical at all in his sense of marks, marking, and school. For him, school was a system that worked well, and marks were a part of that system. Martin appreciated having the teacher prescribe the elements of any task that were of greatest value, and appreciated the feedback that marking represented. Martin especially appreciated having his ability and efforts acknowledged. However, Martin clearly found the inquiry approach to learning in small groups very appealing. In the exchange below, I shared with him the dilemma I faced, attempting to align a very traditional approach to marking with this style of learning. Martin's ongoing shifts in orientation regarding marks provides an opportunity to consider the effects of marks in students' epistemological orientations.

> Martin, October 20--*Homework: What mark would you give yourself for the first 3 topics? Why?*
>
> I would give myself 85% because I worked well in the group, and I didn't always find an answer but I help find the solution.
>
> *You have chosen two specific factors to generate your score, Martin. Working well in the group is very important to the class, and to the group, and perhaps to your success. Should it be rewarded by marks? I wonder how I could measure that factor.*
>
> *You also think that finding the solution instead of the answer is very significant. On a test, how would I write a question that would let you show me your skill at this procedure?*
>
> *Thanks for the intriguing response.*

It was report card time. A considerable amount of the formal discourse between teachers and students and among students concerned marks, but this formal discourse did not inspect the tenuous relationship between marks and learning or the not-so-veiled role which marks played in terms of teacher authority and student autonomy. I was hoping that if the discourse in the homework stayed with each student's themes, they could be encouraged to think about marks in ways that would expose to them their underlying goals and priorities, and thus encourage them to take action toward those goals directly, rather than toward whatever a marking system defined for them as priorities. Martin took advantage of the opportunity, as we returned on occasion to a discussion of how to mark the learning which he and I knew was excellent. In the next exchange, Martin suggests that the teacher is not the best judge for significant elements of the learning process, and proposes a non-traditional approach to testing to notice process rather than product.

> Martin, November 5--*Homework: Answer my response to your marks comment.*
> Dear Mr. Mason,
>
> First I would like to answer your first question. After thinking for a while I believe that working in groups should be rewarded by marks. In the job market and in jobs you will probably have colleagues. You must learn to work side by side with these people or you may be fired. In response to question two I believe that you could measure this factor by getting the people in our groups to evaluate each other. I believe that these people's evaluation should count for more than the teachers' evaluation of our group. Since the people in our groups work each day with each other they will be the best evaluators!
>
> On a test you could ask a question that involves thinking and solution finding, but don't ask for the answer ask for how you would solve the problem!
>
> *Very thoughtful answer, Martin. I hope the time this took you has helped you understand what matters to your learning.*
> *1. Yes, colleagues are essential in most jobs. Learning how to work with specific people is important, especially for people with good ideas. This is a good reason to mark each member of a group.*
> *2. Working together on tough stuff requires colleagues to support each other, and build on each others' strengths. When colleagues evaluate each other, they start noticing each others' weaknesses, and the group suffers. Also, if they must compete for marks with each other, it is harder to be supportive.*
> *Yet I agree with you that if we want accurate marking of each member's group contribution, the group's opinion would be more valid than the teacher's.*
> *3. I like your test idea, Martin. It sure is a different kind of question from most*

math tests, isn't it? Why don't tests have questions like this more often?

Marks was a topic with which I found it very difficult to initiate and maintain meaningful exchanges. In part, this could be because marks was a difficult topic, formal and well-defined on the surface but deeply connected to various personal and school issues underneath. This reason, however, also made marks very much worth talking about. I was more concerned about a second reason for conversations about marks between me and the students being difficult to develop.

Marks were the thin edge of the authority relationship between students and teachers, which meant that whenever the topic was broached between me and a student, our roles and the authority structures which underlay them became more apparent. That doesn't foster risk-taking in writing between a student and a teacher. As the above reply might show, I tried in my replies to be supportive of whatever self-expression students made but still say something worth reading. More important, however, I tried to ask prompts in this area that were more open, to let the student narrow the conversation first. Although this tended to trigger broader and shorter responses, the response would give me something where I could take the first risk. In my reply I would try to give the student something he/she could dissect, rather than have his/her words being the object of our attention. In the following two-stage exchange, Martin first responds briefly to an open invitation to discusses marks with me. Then Martin chooses to build on the extended reply I provided to him. Here is the first response with my extended reply. For Martin, talking about marks is a way to conceptualize what counts as learning.

> Martin, January 28--*Homework: Prepare an idea for me to respond to: marking.*
> Marks? I believe marks are used to rate us on how well we do on notes tests and assignments. They are also used on our report cards to show how much of the course we understand.
> *Suppose someone in grade seven knew 80% of the math. In grade eight, he learned 80% of the math that he could learn, but couldn't get the math based on the grade seven stuff he missed. Then that happens again in grade nine! He'd be getting 80% BUT...*
> *What's wrong with this story?*

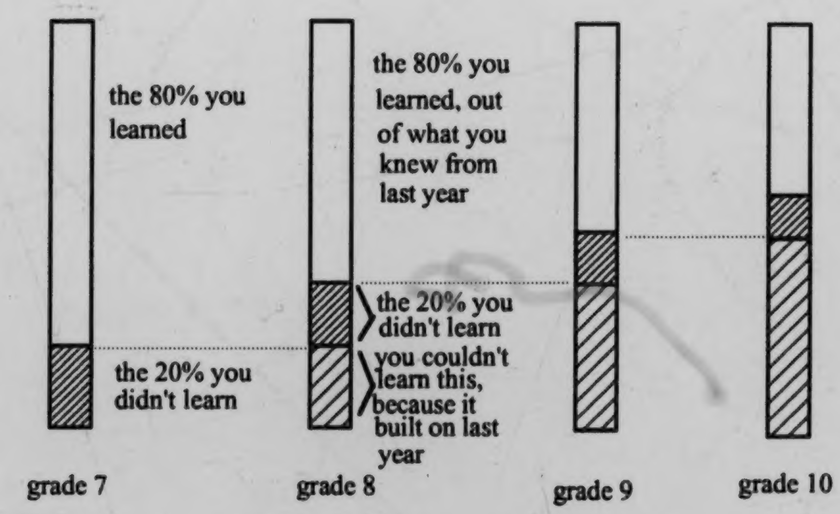

the 80% you learned
the 20% you didn't learn
the 80% you learned, out of what you knew from last year
the 20% you didn't learn
you couldn't learn this, because it built on last year
grade 7
grade 8
grade 9
grade 10

Figure 6-1. Part of my January 28 response to Martin.

Martin shared the above reply with his group. They talked at length about it, although only some of the conversation came to my ears through the classroom tape recorders. At one point, Wai said, "It's sort of like compound interest," but neither of his group-mates took on the comparison. For me, it was more like a model of radioactive decay, but I didn't realize I was working with a metaphor at all until I heard Wai's interpretation on tape. Regardless, I have added this idea to my pedagogic examples of exponential functions.

In part, this reply represents a deep commitment to the necessarily dialogic nature of response journals. Too often, teachers using journals view their role as being simply the provision of prompts, reading the responses, and writing a brief encouraging reply in the margin. However, if such structures are going to be used to scaffold students' engagement with difficult issues such as identifying and pursuing their own goals as learners, then I think teachers may have to accept a more involved role. Often, such replies will fail to elicit more detailed responses, but on the occasions when students intensify their involvement with the issue and with the discourse, the rewards are significant.

As Martin accepts my previous reply as his prompt for a further response, the homework structure becomes more dialogic. More important, Martin builds on his own sense that marks show both "how well we do on notes tests and assignments" and "how much ... we understand." The reply I had given him gave 80% a numerical meaning, that of learning 80% of the material. The idea of missing out on 20% of a course's content and still getting a strong mark didn't work for Martin's sense of what 80% meant as a school mark. Nor did the idea of the amount learned at 80% per year decreasingly drastically each year. To reconcile the ideas, he thought about what might happen in school. Below, Martin uses the visual format I provided to express his understanding of how 80% can be maintained over a student's career.

> Martin, February 3--*Homework: Respond to any one idea from last week.*
>
> If the student doesn't know 20% of the grade seven work he will probably cover the topics again in grade eight and probably master them. If he receives 80% in grade 8 he won't know 20% of the grade 8 concepts. In grade 9 he will go over his grade 8 problems and if he gets 80% then he won't know 20% of the grade nine concepts. This way he loses 20% each year . The next year he gains what he lost but loses some more. [See Figure 6-2, below.]

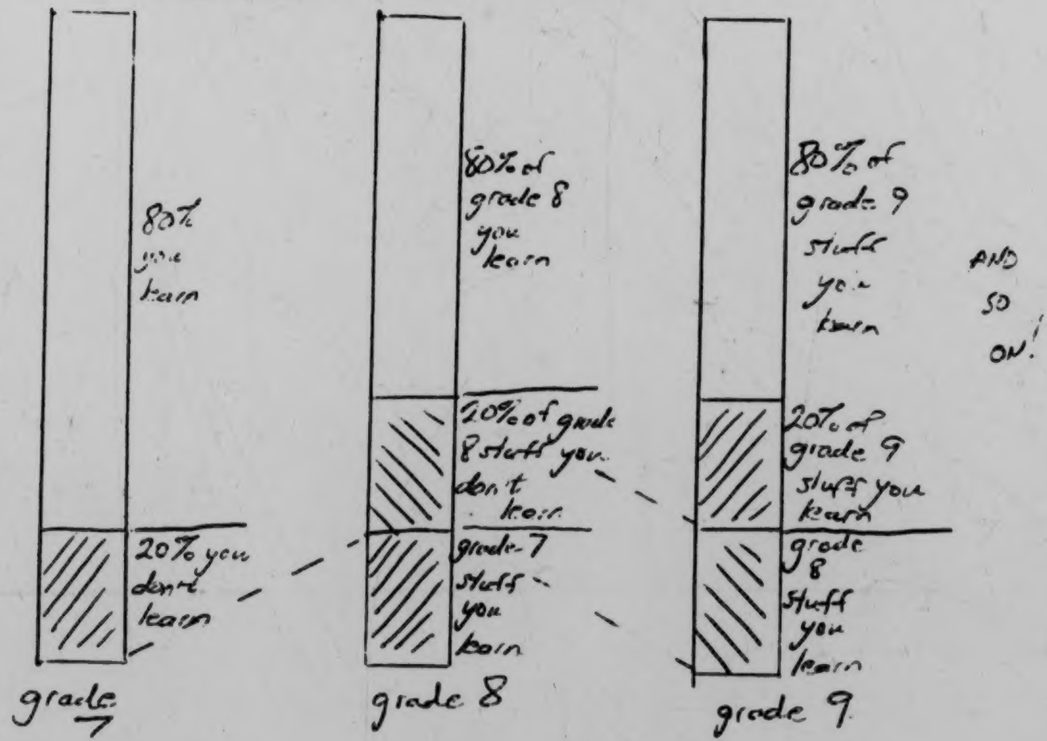

80% you learn
20% you don't learn
grade 7
80% of grade 8 you learn
20% of grade 8 stuff you don't learn
grade 7 stuff you learn
grade 8
80% of grade 9 stuff you learn
20% of grade 9 stuff you learn
grade 8 stuff you learn
grade 9
AND SO ON!

Figure 6-2. Part of Martin's February 3 homework.

This can go on until you stop taking math courses!

> *Brilliant, Martin. Your diagram and words are a great combination to explain your thinking. I think I agree that kids who get 80% probably can pick up the missing stuff during the next year. Anyway, kids who get 80% probably knew 90% of the stuff, and understood 70%, so they're in good shape.*
>
> *What happens to kids with 65%? I don't think they can make up all they missed, the next year.. Each year they might make up 25 out of the 35 ideas they missed, but that leaves a gap of 10 or more, each year, doesn't it? What happens after 3 years of that? Try my diagram, and a version of yours, to see what the possibilities might be.*
>
> *What do you do that helps you pick up the stuff that you missed the year before? How will you change (or improve) those processes next year, when there is much less review and a much faster pace?*

If Martin accepted my invitation to do another version of the diagrams, he did not share it. I seldom know in replying to students when to accept a response as the terminus of a flow of ideas and when to point further. Maybe my task should be to point further each time, and leave the student free to choose whether to accept the invitation or not. In this case, although Martin did not respond directly, he maintained a high-energy approach to the homework exchanges, and continued to consider marks and learning in different ways. The next example shows the ongoing progress Martin made in conceiving learning as a complex process.

Describing learning without marks

This response followed a special event. To close the algebra tiles inquiry (Chapter 1), students wrote a "test" in which they explored further with algebra tiles. In a way, it was a test true to Martin's suggestion to value the search for solutions more than right answers. In part one, students on their own explored how to complete the trinomial $x^2 + \_\_\_x + 16$, and then started an exploration of trinomials that made "perfect squares." After part one ended, each student was paired with a student from a different learning group to discuss what they had done on the test. Part two created no paperflow but at the end of the period each student was invited to write to their partner about his/her learning. I was very pleased with the results: students capably engaged with the ideas of learning and tests in various complex ways.

Partly to offer a baseline for Martin's writing about his partner, here is Wai's writing about Kelsey. There is reference to some specific content, some admiration expressed for Kelsey's ability to say things completely, and a reference to "ways of seeing things on how to do math" that, unfortunately for the purposes of this chapter, are not developed. After my reply to Wai's response, Kelsey has added a reply too. Wai responded to that, and I closed what might otherwise have been an infinite cycle. This passage further illustrates how the students came to perceive their learning in a more complex way within the context of the inquiry program.

> Wai and Kelsey, March 3, 4--*Homework: Say something about your partner's learning. Respond to their comment.*
>
> [Wai] My partner's learning was great! Kelsey showed me a different way of doing things, for example: In the test, I never really thought of using negatives, but she showed me how. That was a spark of creativity and it turned me away from just using positive numbers. Her answers were also quite complete and gave the test flavor and they were all right! She also gave me more ideas and ways of seeing things on how to do math.
>
> *I am not surprised that you two did as well together. Normally it is very difficult to work with a new partner on complicated stuff. However, you guys have both learned to value complicated ideas. You both have been practicing being supportive of group members as you learn together. Wai, what she "gave" you is just what you wanted. Your wanting it was the key. I would enjoy learning more about this.*
>
> [Kelsey] Wai really helped me see doing the questions in a different way. I saw the way he did things, and tried them. Working in partners really helps because you see things in different ways. Kelsey
>
> [Wai] Thanks, I'm glad I could help you see things in a different way. And I'm glad you tried these new answers out. Thanks for the open mind. Wai
>
> *I'm glad you both value seeing more than one way to get an answer! I'm glad you both value interacting with others as a learning procedure. Do you perceive this as indicative of you tremendous progress this year? I do. Want to talk about this?*
>
> *P.S. How would I give you a mark for this kind of progress?*

Wai and Kelsey represent well the rest of the class, in regard to everyone's recognition of the importance of being confirmatory in discussing a classmate's test. However, they also represent well how everyone found things which were unique about their new partner's approach to the same questions which they had done. None of their classmates made any attempt to give a numerical score in response to this invitation to talk about their classmate's learning: that is solely the teacher's responsibility, or burden. Instead, as Wai did, they described the learning, first with a general positive description, and then with details about what impressed them. Wai and Kelsey's mutual respect for "doing the questions in a different way" suggests a very different epistemology from the traditional view that what matters is knowing the best way, the right way, the teacher's way, to do a particular question.

Ivan's responses to part B of the test may look a little disappointing, in comparison to the more complex and interactive exchanges above. Indeed, his initial response did disappoint his partner for the activity, Hilde. Perhaps part of that may have been Ivan's discomfort at being paired with a very confident female, but, as the written discourse makes apparent, there is also a strong contrast in purposefulness regarding this task and in what counts as learning mathematics generally. Hilde's reply to Ivan's original response is followed by my mediatory reply to both. Immediately following that is Hilde's response to Ivan, and his somewhat sheepish response. My reply to that exchange is a recognition that Ivan has been learning about learning, but admittedly has more to do.

> Ivan and Hilde, March 3, 4--*Homework: Say something about your partner's learning. Respond to their comment.*
>
> [Ivan] My partner Hilde's learning is excellent because she has a good understanding of the algebra tiles and she knows how to use the grid multiplying method. She also used square roots as a pattern to help her do good quality work.
> [Hilde] Ivan, thank you very much for the compliments, but what did I not do good? I know I can think of some things that I could have elaborated on, especially the first question! I'm glad you were able to see the patterns I saw (because someone else might have thought I was crazy! Ha Ha!
>
> P. S. Ivan don't be angry with my comments, I thought we were supposed to write on what could be improved upon! You did well and I hope you know I think that! Hilde
>
> *Hilde, your concern for Ivan's confidence is admirable. It is very difficult to offer observations that are meaningful, and at the same time offer the support people always need. This is the teacher's dilemma! Do they just offer support and compliments, or do they also offer honest suggestions for improvement? This test*

was another attempt to do both, perfectly. That's impossible, of course, but I refuse to chicken out on you.

Ivan, I am impressed with your description of what you noticed with Hilde. You mention some important math ideas, and you let Hilde know how you felt about them. Hilde has let you know how eager she is to learn and to improve! She has suggested that she would value your perspective on this, and that's an honor.

[Hilde] Ivan did pretty good on his test. He answered all the questions but I think he could have expanded on some of them. Ivan thought of some things I did not think of and explained his work in a somewhat similar way as I did. However, I know that more things could have been done especially on the first question. All in all, he did very good, I think he has a good understanding of the concepts taught.

[Ivan] Thanks for making it sound like I learned something, Hilde.

I agree that Ivan has a good basic understanding, Hilde, and listening to him is a good way to notice that. His paperwork and his group work do not show that understanding as well as a conversation with him does. In general, he likes to get things finished, so doing MORE with a question doesn't appeal to him. Yet! (He's still learning, just like we all are.)

Now we can switch out attention to Martin's response to Part B of the test, in which he and Maria had shared and discussed their responses to the test. Maria was introduced in Chapter 4, and Chapter 7 will conclude with Maria's response about Martin's learning. However, for now, the focus will be on Martin, and his developing epistemology. Martin uses his personal preference for diagrams to spontaneously organize a template for inquiry-based learning in small groups. .

Martin, March 3, 4--*Homework: Say something about your partner's learning. Respond to their comment.*

Evaluation of Maria
Maria has a very unique way of explaining what she did (in the aspect of learning) to achieve a solution. Maria has very good ideas which she can convert into examples, grid multiplying, and drawings. She gives many examples for each solution to show her understanding. She has a very good understanding of what she talked about on the test and she showed this through her work as well as her talking.

What her talking led to

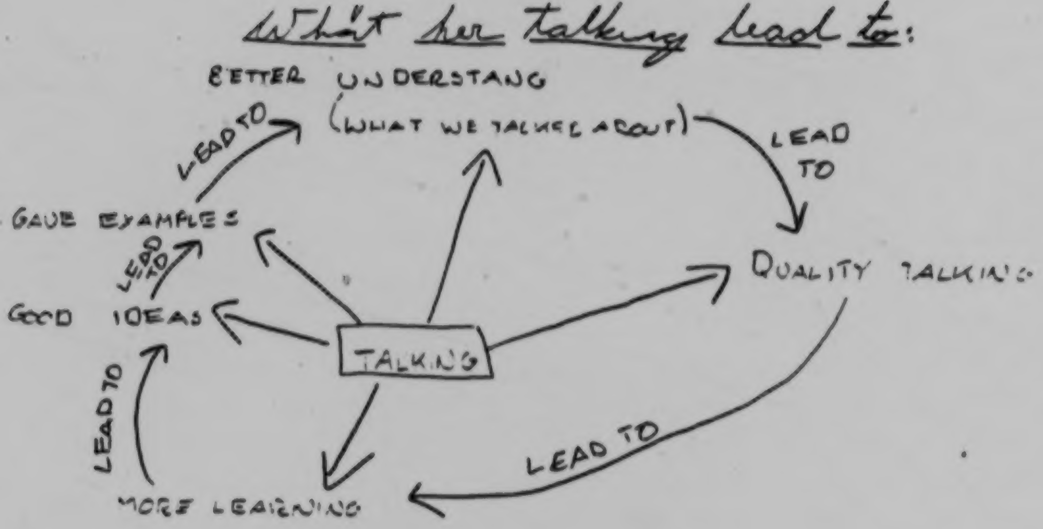

What her talking lead to:
BETTER UNDERSTANG
(WHAT WE TALKED ABOUT)
LEAD TO
LEAD TO
GAVE EXAMPLES
LEAD TO
QUALITY TALKING
GOOD IDEAS
TALKING
LEAD TO
LEAD TO
MORE LEARNING

Figure 6-3. Part of Martin's March 3-4 reflection.

Wow! This blows me away! I can't think of how to respond. Please talk to me about this. I want to learn.

Clearly, Martin had been learning about learning. In describing Maria's learning in words, he recognized the value in expressing ideas in multiple formats. He gave special status to the role of examples for showing what she understands. In the diagram, he gave a further role to good examples: not only do they emanate from good ideas, they lead to further understanding. Martin later clarified that the I as teacher could provide "quality talking" or "good ideas" or "gave examples," or the students could provide it all if they were dealing with a good enough activity. The "more learning" was what the students did in direct response to quality talk. Then the "good ideas" and "gave examples" were making understanding, and finally the "better understanding" was what the learners ended up with and used to begin all over again. He felt that "quality talking" had to rely on students understanding what they already had learned, so the cycle continued. I wish now that I had asked him where writing homework back and forth fit in with all of this, and I wish I had asked him where drawing (his own preferred format) fit in, but the richness of what was included surpasses any possible additions. Martin was expressing a deeper understanding of how students come to understand in a classroom oriented by the teacher in a constructivist way. The meanings of experiences were determined by the form of the interactive reflection which he was guided to do: they were meanings he had constructed himself to describe to himself the meanings of his activities and interactions with students and with me.

Student epistemology: the starting points

> My students will begin to view mathematics as a sense-making discipline, something about which to think rather than to guess (Andrews, 1995, p. 152).
>
> Students [must] *become aware of and reflect on their beliefs, as well as possible alternatives,* since beliefs are more powerful the more they are held unconscious and unquestioned. ... It is important therefore to offer students opportunities to engage in mathematical activities that can generate doubt in assumptions taken for granted up to that point (Borasi, 1990, p. 179).
>
> Vision and strategic planning come later. ... First, under conditions of dynamic complexity, people need a good deal of *reflective experience* before they can form a plausible vision. Vision emerges from more than it precedes action (Fullan, 1993, p. 127).

Does the data in this chapter illuminate the nature of the students' learning about learning? The rich and personal nature of learning as an ongoing process cannot be represented without such long and thick descriptions. A harder question for me would be the traditionally simpler version of that question: how much did Martin learn? This kind of data does not lend itself to quantification, but I would suggest that learning itself does not lend itself well to quantification, either. Lately, I have taken to answering the question, "How are you?" with numerical scores out of ten. People are amused, and enjoy replying to my responses, but we seldom broach the irony of me quantifying my response to a question that lends itself as poorly to such answers as it does to the brief ones we traditionally use.

What would it mean to attempt to quantify Martin's learning about learning? Consider it perhaps to be a challenge for a pre-post design, and consider where Martin began. At the beginning of the year, I asked the homework question, "Three things that make me a good math student." As you read Martin's response, I think you will recognize that Martin progressed considerably from where he began. However, I also trust that you will recognize that was is significant in that progress is not its (quantitative) extent but its particular (qualitative) nature.

> Martin, September 11--*homework: Three things that make me a good math student.*
>
> Me Personally
>
> I'm very patient and I have the ability to sit there and listen to a concept over and over again till I catch on. I have the ability to do my math homework quickly and correctly. I ask many questions in math. It will allow me to clear up problems for myself and others.
>
> *I am glad you understand that 'catching on' is more than just hearing a concept. You are right that one person's question can help many classmates. An expert clears up problems by asking himself questions but that is best learned with others to ask and to answer!*
>
> *Thanks for saying you liked some of Monday's class!*

The next day, I asked the students to describe themselves as math students.

> Martin, September 14--*Homework: What makes you a good math student?*
>
> I am very patient. If I do not understand something, I can sit there have it retaught to me without getting upset or frustrated. I ask many questions during class which allows me and other students to benefit from my questions and also the answers. I have the ability to do my math work independently and also if chosen I work very well in a group. I grasp on to math concepts quite quickly which allows me to help others around me if the teacher is busy.
>
> *I like very much what you wrote, Martin. I picked two contrasts out. Your patience is an asset that you have seldom had to use, if you usually grasp math ideas quickly. We will be using your patience more, because we will be pursuing more complicated ideas. However, don't just sit there patiently. Asking and thinking and guessing and checking are ways to be aggressive and succeed while you are being patient.*

At first glance, this writing makes Martin appear to have been simply a passive or receptive or compliant learner. Martin described teaching as a matter of explaining and repeating, and learning was sitting and working and grasping concepts quickly. Martin felt that catching on to concepts quickly was commendable, but if more time was needed then Martin felt that passive patience was commendable too. Yet Martin reported asking many questions, which may not seem to fit the traditional image of a passive learner. This is quite in keeping with compliance in the typical participatory-explaining style of most teachers, however, where student input consisting of answers to straightforward questions and questions about the examples being discussed is part of the process. The questions Martin would have asked are the questions a teacher would want asked, as invitations to explain "over and over again". Clearly differences are clear between what was expressed early in the year and what Martin described as good learning in his later writings.

As a side issue, I think it is quite clear too that as the research strand began I was only beginning to understand how to respond to the students' writing each day. Martin wanted me to say Wise and Helpful Things, and, like Owl for Winnie the Pooh, I was too willing to pontificate (Milne, 1926). For instance, you can see in my second response above that I both judged and advised in response to Martin's statement about sitting passively: I learned that judging and advising were too direct an attempt to influence to accomplish my goals. Judging and advising were too aggressive to sponsor further deepening of the discourse as well, but I learned that sort of thing as I went along. I do not feel guilty about this: I must remember that these documents were only one element of the relationship, and cannot be a record of all that we said to each other. More important, it takes time for personal reactions to become interpersonal interactions. That this came to pass suggests that these early attempts, clumsy as they might have been, were effective. They were honest and eager and personal, and even my advising was evidence to Martin that I had listened to his words. I think that over time I learned to consider the particular student more than the particular point I felt like making. Beyond such descriptive claims I could not quantify my own learning in this regard any more than I could Martin's.

In the beginning of the research Wai and Ivan were also feeling their way with a new style of interacting with a teacher. On the second of the two days from which these samples are chosen, they tested the waters regarding being able to write as little as they wished. I must be careful not to read too much into why they chose not to engage when the question was so personally evaluative. On the first day, however, they both give illuminating answers. Although Wai, like Martin, is glad to be working in groups instead of alone, the rest of his answer addresses the fundamentals of being a good student. I do not think he is oversimplifying in his description of what problem solving means to him within the context of math class, unfortunately. Ivan, on the other hand, is certainly keeping things simple. He wrote an answer that responded directly to the question, and the question hadn't said the answer had to be a full sentence! Again, the nature of Wai and Ivan's epistemological progress is particular to the individual, and is not conducive to being measured.

> Wai, September 11--*Homework: Three things that make me a good math student.*
>
> I put these three things on my list, Organized, understand how to do problems, and work in groups. I am organized, I always bring my books, pencils, pens, calculators and equipment to every math class. I understand how to do problems. Like how to add or multiply, subtract or divide, how to do equations, how to do number problems. I also like to work in groups, it makes doing work and figuring out problems fun and easy.
>
> *Not many people mentioned working in groups, Wai. We'll be doing it a lot, and I'm glad you feel good about it. It's more than just a matter of fun and easy, though. working in groups lets us tackle more complex learning issues, and it also gives us a place where we can control tough feelings, like you mentioned on Monday.*

> Ivan, September 11--*Homework: Three things that make me a good math student.*
>
> patient, pay attention, communication.
>
> *On Monday, it was your patience that was put to the test, judging from what you wrote.*

But is this epistemology?

Should we categorize these learners in relation to their prior understanding of what learning is about? With the responses of all three students side by side, it is possible to see the rich diversity that would still exist within whatever categories we might create for them (Baxter Magolda, 1992). Although similarities exist across their answers, and thus similarities could be claimed to exist across their epistemologies, distinctions call out to be noticed. At the level at which a teacher must operate when helping these students to learn epistemologically, the similarities cannot guide choices sufficiently. Similarly, the personal learning which was evident in the ongoing responses earlier in the chapter has richness which defies generalization, and that richness must be central to the interactions of a teacher with these particular students to foster continuing change. The significance of the particularities of students to efforts to truly understand or affect their learning is why the categories which I had intended to use in this research (Belenky, Clinchy, Goldberger, & Tarule, 1986) fell away.

I am not repudiating the categories which clarified ways of knowing in this dismissal of their use, however. As the students learned, and as they learned more and more about their own learning, their ways of knowing clearly changed. Perhaps it is the ultimate validation of a model that as it illuminates particular details in learners' growth, other salient elements which the model does not address (in this case, particular personal distinctions) are not hidden from view but rather stand out because they don't fit within the model. My claim here is that for anyone attempting to interact meaningfully with students as they change their ways of knowing, categorizing them is insufficient analysis to guide those interactions. On the other hand, a model of categories can provide a lattice for structuring the interactions and the growth that does occur, and for analyzing the growth.

I have suggested that it is reasonable to address a question such as, "Did this student learn about learning?" or even "What has this student learned about learning?" through a retrospective comparison of starting points and endpoints. Perhaps this is the proper role to be assigned to summative evaluation processes. However, it is inappropriate to ask of such processes the slight variations on these questions which teachers must ask themselves over and over again while the students are learning. Because teachers care about the student, and because their job is to cause learning to happen, the question they must ask and ask is, "How is this student learning about learning?" Unlike the two questions which lead this paragraph, this new question requests qualities regarding a process, not quantity in relation to a product--it asks *how*, not *how much*. It is also a present-tense question, rather than past tense, for two inter-relating reasons. First, to achieve significant learning about learning, that student must benefit from the teacher's developing understanding about her, and so the teacher must ask the question early, and often. To the extent that the teacher's inquiry processes are open to the learner, the learner may also benefit from her participation in the processes, as Martin, Ivan, and Wai benefited by answering the homework questions. Second, the question is a complex one influenced greatly by the idiosyncrasies of its subjects, and such complexity can only unfold over time, as and with the unfolding of the relationship between student and teacher.

This chapter presented as data the residue of a process which was pedagogic in intent, not only to ask "How is this student learning about learning?" but also to help both the relationship and the learning unfold in its complex ways particular to each student. The smaller-scale labels currently used for such learning, such as meta-cognition and conation (Keefe & Walberg, 1992), seem limiting for this purpose. The students learned more complex ways to learn, but they did not do so by learning about learning per se. They learned math in different ways, and came to know learning in a richer way. They discussed student concerns such as homework and evaluation and marks, and came to know learning in a richer way. However, they did not remove learning from its contexts of mathematics and school to make it the object of their attention. Their understanding of learning and knowing was derived from reflecting on their personal inquiry, but it was not the object of that inquiry, and thus it is difficult for me to call this process metacognition. They learned about their own learning, not about learning as a disembodied object of study.

Is Martin's learning epistemological? It involves and reflects his sense of how learning goes, his sense of how understanding happens and what understanding is (Bateson & Bateson, 1987). To say that this is not epistemology would be to say that the word is reserved only for philosophers playing a closed and static game. Epistemology has been criticized for attempting to be about knowledge independent of knowers (Cobb, 1995c; Freire & Macedo, 1995; Greene, 1994). Martin and his colleagues provide a personal and interpersonal framing for the knowing and the learning about knowing, and so illuminate their epistemology as an element that is alive as it develops concurrently with its host and its host's context.

Should this research count as epistemological inquiry? When inquiring about the origins of the universe or the forces within a black hole, it may well be necessary to build that inquiry on a search for extensions of generalizations and to test those extensions for contradictions. Direct evidence for the subject of study is not at hand, and interacting with the subject of study is impossible (Hawking, 1988). The exact opposite is true for epistemology as a field of study. Both learning and knowing are processes which are not only readily at hand, but relatively open to our interactions and influence. To call the research described in this chapter epistemological inquiry represents a shift in what would count as epistemological inquiry from the challenge of constructing structures that can describe knowledge as a human product to the challenge of understanding and influencing the interpersonal processes of coming to know. Such epistemological inquiry determines its truth value not through generalization without contradiction, but through viability in given contexts and for particular persons of their working understandings.

Above all else, Martin's data makes clear that epistemological stances change/

mature/grow. This growth happened in relation to activities which allowed Martin to stretch his approaches to learning, and activities that brought learning (his own and others') into his view. In other words, epistemology is learned, and to the extent that we view teaching as the influencing of learning, it can be taught. This conclusion is a prerequisite to a stronger one: for learning to grow in ways that less haphazardly support learners as they face new challenges, epistemology must be taught. This definitely does not mean that a particular epistemology must be taught, or that a particular developmental pathway must be targeted as a curriculum. Rather, epistemological growth must be a targeted component of each student's learning within curricula constructed from a constructivist perspective. Such a translation of epistemological theory to constructivist pedagogy suggests a marriage of the two more than a divorcing (Cobb, 1995c; Noddings, 1990; von Glasersfeld, 1991a).

It is the nature of this claim and the logical consequence of this research method that the voices of Martin, Wai, and Ivan provided the evidence to support the claim that the changes they underwent are epistemological. Yet the question was not posed in a vacuum, and some readers may wish to frame the question within academic discourse. To that end, the last word goes to other educators' claims about epistemology. The selections present a disparate front, suggesting that what counts as epistemology within educational discourse is open to questions and to being shaped by its further use such as within this chapter. Rather than risk submerging under the weight of philosophical claims the clear but soft voices of Martin, Wai, and Ivan which portrayed their evolving senses of what it means to learn and to understand, the quotes are left unembellished with interpretation.

> Epistemology is an indivisible, integrated meta-science whose subject matter is the world of evolution, thought, adaptation, embryology, and genetics – the science of mind in the widest sense of the word. ... Epistemology is the bonus from combining insights from all these separate genetic sciences. But epistemology is always *personal* (Bateson, 1979, p. 87).

> For many philosophers, an endorsement of relativism signals the end of knowledge and epistemology (Code, 1991, p. 2).

> We view standpoint epistemology as a way to make explicit the point from which inquiry begins; unlike Cartesian epistemology, in which the knowing subject comes from nowhere, the epistemic subject of feminism provides a view from where she is (Currie, 1992, p. 355).

The problem of knowledge as conceived in the industry of epistemology is the problem of knowledge *in general*.. ... But there is no problem of epistemology in general. ... Knowledge is one way in which natural energies cooperate (Dewey, 1960, pp. 41, 42, 43).

As recently as a decade ago, epistemologists almost never emphasized the social features of knowledge--indeed, to do so would have been to violate the sacrosanct, since it was understood that the goal of epistemology was to give an account consonant with truth and the avoidance of error, none of which has anything to do with the social aspects of knowledge acquisition. ... Bluntly, "epistemology" in disciplines other than philosophy, has come to mean "ways of knowing" (Duran, 1991, pp. 48, 80).

If, for analytical purposes, we distinguish between an "epistemological" and a "mathematical" strand of the teacher development process, we can say that we have had no theory of the mathematical strand or of its relation to the epistemological. For us, the concept of teaching for the construction of the big ideas now poses the challenge of theorizing the unity of the two strands as a single complex process (Schifter & Fosnot, 1993, p. 191).

In a truly relational epistemology, however, the knowing subject is active as a determinedly intentional, meaning-seeking body (Sorri & Gill, 1989, p. 34).

Evolutionary epistemology does not reduce humans to machines or heaps of atoms, but rather helps to explain their *complexity* as result of complex evolutionary processes (Wuketits, 1990, 206).

Chapter 7. Assessment as an Ontological Construction Zone

This chapter's purpose is to propose and demonstrate the usefulness of considering students' ontologies to be a vital element of students' interactions with mathematics education reform efforts. "Ontology" is used here as a label for students' individual senses of their being and their becoming, including their sense of role, sense of personal purpose, and actions as people, as students, and as learners in the contexts in which they act.

> Through complementary acts of naming and framing, the practitioner selects things for attention and organizes them, guided by an appreciation of the situation that gives it coherence and sets a direction for action. So problem setting is an ontological process -- in Nelson Goodman's (1978) memorable word, a form of *worldmaking* (Schon, 1988, p. 4).

> Cognition in its most encompassing sense consists in the enactment or bringing forth of a world by a viable history of structural coupling. ... Thus cognition as embodied action both poses the problems and specifies those paths that must be tread or laid down for their solution (Varela, Thompson, & Rosch, 1993, p. 205).

Such a complicated element is a challenge to label. Rather than engage in verbipoeisis (literally, the genesis of new words, such as my use of "verbipoeisis"), my choice involves a shift in the use of a philosophical term, ontology, paralleling the previous chapter's use of "epistemology." Other possibilities included "role" (Berger & Luckmann, 1967) and "self-concept" (Kohn, 1994) but each of these placed inappropriate boundaries on the element. The chapter will show that students' involvement with reorienting their learning of mathematics is more than just role-deep. Students did not view the changes described here as being only about themselves within the role of student within the context of math class or school: these changes affected and were affected by their broader identity as unique individuals in interaction with others. Similarly, ontological learning may not be bounded solely by changes in self-concept (how one perceives oneself): when learners change how they learn, they are changing what they are and what they do, not just how they perceive what they are. "Ontological learning" suggests the deeply personal nature of the changes which this chapter describes without bounding what the label may come to signify when particular cases are viewed.

Ontological learning will not be presented as changes that can be viewed or caused in isolation from the learning described in previous chapters. The students' learning about and changing their sense of themselves was concurrent with and interdependent with their learning mathematics and their learning broader epistemological senses of mathematics and learning. As a consequence, this chapter proposes considerations of ontology as a way to enrich how previous chapters have suggested students' learning be understood, complementing and interacting with the themes of those chapters. Ontological changes will be viewed within the contexts in which they are naturally embedded, and the use of students' own words will help to reflect the personal and interpersonal aspect of this facet of learning. Because ontology will be presented as a dynamic realm where learning and teaching occur, the premises for viewing learning from a constructivist perspective (Chapter 3) will help illuminate the processes that are involved.

Put in simpler terms, this chapter is just about what students do when they learn and what they learn about what they do. Throughout the chapter the intended implication is that the students are asking the unvoiced questions, "Who am I?" and "Who am I becoming?" as they make choices about how to conduct themselves and reflect on their experiences.

Although it seems a tautology, it is significant that these students see themselves as students. In the context of the math class, they see themselves as successful people largely as they see themselves as succeeding at studenting, the performing of the tasks which math teachers prescribe. However, the behaviors which make for successful inquiry are more complex, more ambiguous, and more personal than the behaviors expected of students in traditional pedagogy. "Problem-posing education affirms men [sic] as beings in the process of becoming--as unfinished, uncompleted beings in and with a likewise unfinished reality" (Freire, 1970, p. 72). Thus, students encountering an inquiry-oriented mathematics program for the first time must reconstruct their image of themselves as successful learners. All this is on top of (or embedded with) the learning of the algebra and the learning of the group inquiry skills which were highlighted in Chapters 1, 3, and 4, and on top of (or embedded with) the epistemological learning highlighted in Chapter 6. It makes sense that people engaged in learning and understanding would see themselves differently as their senses of how they learn and what counts as understanding becomes more complex. The chapter will present evidence that students' ontologies are relevant to their learning of mathematics (curriculum) and their learning about learning and understanding (epistemology). Further, students' ontological senses of themselves are subject to their re-construction. It is learnable and it is teachable, concurrent with the learning and teaching of mathematics and epistemology.

This chapter features Maria, whom Chapter 5 showed with her three colleagues engaging successfully with the challenges of Rectangles Families. Maria is a wonderful guide for the purposes of this chapter. She expressed herself openly even while she was uncertain about herself. Within that uncertainty Maria wanted to develop an image of herself in which she could be more confident. As a result she was open to changing her perspective, making constant use of the "mirrors" available in the classroom, including her classmates with whom she compared herself and the feedback from teachers which came from assessment processes. Faithful to Maria's approach, we will look to other students also to help distinguish aspects of Maria's ontological changes. Also true to Maria's approach, assessment as an element of the inquiry mathematics program will receive considerable attention.

Tests and confidence

> Like the concept of zero in mathematics, a concept of self is pivotal in organizing experience, useful as an idea as long as it is not mistaken for a thing. ... The self is learned, yet ironically it often becomes a barrier to learning. The illusion of autonomy confers a sort of immunity, often tenaciously defended, to the effects of new contexts and relationships, yet in order to move through society, we are asked to put the tenuous certainties of the self at risk again and again. ... Even when we try to build up the self, we subvert it for the sake of discipline and conformity. It is almost as if schools demanded, Leave your self, your self-esteem, the confidence accrued from learning to walk and speak, at the door (Bateson, 1994, p. 67).

The first portion of transcript follows directly from Chapter 4. In fact, it is a continuation of the portion of transcript which closed that chapter. In that context, the students' discourse was used as an exemplar of small-group inquiry. Complex ideas were being developed with everyone's involvement and those ideas were expressed in multiple ways. During all this, the group simultaneously managed the inquiry through discourse aiming for consensus. Yet Maria's words do not necessarily show that she was feeling the confidence which her group-mates expressed.

Maria was a very capable student and generally she was well aware of her capabilities, as the next portion of transcript will show. Yet she was in a learning group with three students who were more self-assured than she, especially when inquiry involved making conceptual leaps that were not carefully explained. Maria offered a special role for the group, in that she often insisted (and, in the Chapter 4 transcripts, was invited by her group-mates to do so) that the group fill in the details of those leaps after they were made. In my view, she had a different sense of rigor which was of great value to her and to her partners. However, Maria felt more and more uncomfortable that perhaps she was not learning as well as the other three. When anyone in the group suggested an effective intuitive leap, usually none of the other three followed the logic of that leap right away. Yet Maria's three colleagues were comfortable with such tenuous steps into uncertain ground, while Maria may have read in their relative comfort that her colleagues had immediately perceived the logic that could explain the proposal as a sensible one. From her more traditional sense of mathematics as a sequential and step-by-step process, her group-mates' degree of comfort with the uncertainty of exploratory inquiry wasn't available to her. This is my interpretation however, more analytical than Maria could provide in her own words for herself or for others. As the transcript below will suggest, Maria felt confident in some respects about her learning but she also felt a general sense of uncertainty, one which she had yet to put into terms which made sense of it for her. The prospect of a test to come made Maria feel vulnerable enough to broach the topic with me, with her colleagues around her for support.

Mr: Mr. Mason, when's our test?
MrM: We'll see, because we have to finish this.
Mr: Okay, sure.
MrM: We'll have to wait till everyone's finished, and then give you a test.
Sv: We don't have a test!?
MrM: No. We do.
Mr: We do? Seriously?
MrM: Seriously.
Sv: On what?
MrM: On this.
Mr: That's a scary thought!
Sv: What are you going to ask us?
MrM: I'm going to ask you to prove to me how well you understand algebra, and how well you have been learning.
Mr: Sometimes some of us,
Mo: So we don't have to use numbers.

Sv: Prove to you!

Mr: Sometimes,

MrM: Prove is too strong a word. Basically you are here to learn some algebra. You're here to learn how to learn. And you're here to succeed in math.

Mr: Okay.

MrM: Your test will be designed to tell me how well you're learning algebra, how well you're learning to learn, and how well you're succeeding in math.

Mr: Some of us have a better understanding of algebra when we work in groups and we discuss it, rather than all of a sudden you give us a question and we're sitting here on our own, you know.

MrM: So the group process is part of your understanding of algebra?

Mr: Yeah.

MrM: Okay. It's also been part of your learning of algebra.

Mr: I think these things are better than a test.

MrM: What?

Mr: Working on problems, like this, rather than tests, because you're in a group,

MrM: So if you learn it in a group, you'd be unprepared to be given a test that isn't in a group.

Mr: No, not necessarily, like a test,

Sv: If you learn it then you know it,

Mr: A test is an on-the-spot thing, you know, you get it right or wrong.

MrM: I don't want to know just what you already know. On a test, that's what most of us do, but that's not what I want.

Mr: Yeah?

MrM: I also want to know how well you learn.

Mr: Yeah.

MrM: So my test has to be designed to be a learning experience for you.

Mr: Do you give us a mark, a percentage?

MrM: That's my job. It's right in the School Act, provincial legislation that governs schools, it tells teachers that,

Mr: And that goes in math challenge, or what does it go under?

MrM: Actually, I think you guys are going to do very well in math challenge, even Hilde who is being quiet here, who is thinking it's all about working and getting answers,

Hl: No I'm not, I was writing this down.

MrM: Super. that's good -- you're fighting back. Now I have faith in you again. [Hilde laughs.] If you think that it makes more sense that the group work is part of your learning,

Mr: Yes!

MrM: And if I think that, then the test has to be a group test.

Maria clearly had a narrow range of possibilities associated with the word *test*. She expressed very clearly her sense of accomplishment regarding the learning that was happening that day in her group, but had no image of a test that could capture that. More significantly to her, she felt threatened by the idea of a traditional test: it removed her from the group context where she felt effective; also a traditional test demanding "right or wrong" answers about content wasn't a match with what and how they had been learning. I agreed with all these thoughts, and expressed as much, but there were huge differences in our stances with this issue. I perceived a much wider range of possibilities for the style and structure of a test than Maria did. Also, I had extensive evidence of Maria's capability as a learner generally and her success with this content in particular, sufficient to assume that if a test didn't indicate Maria's excellence the problem was with the test, not with Maria or with her learning. Maria wasn't yet able to feel such confidence and may not have been able to perceive a test as something so potentially fallible.

Yet these are not the most significant differences in our two perspectives on the test. Whereas for me the challenge of aligning a test with the instruction was an intellectual pedagogic one, for Maria the test's alignment was a question with significant personal ramifications. Maria's belief in the kind of learning she was doing was tentative, and when she looked forward to the test for the confirmation she desired, she worried. If the test betrayed her budding but shaky sense of success with this particular style of learning, it would mean much more than just a lowering of her mark. Maria's previous confidence, confirmed repeatedly over the years by tests from within traditional teaching-learning relationships, did not extend unshakably for her into this new context. A poor test score would suggest to Maria that for this kind of learning she wasn't as talented as she was for more familiar kinds.

Although I am writing at length about this challenge as I perceive it, Maria did not express her misgivings strongly here: vulnerability is not the stuff which make words flow. Also, it was not her style, by which I mean that it did not fit with Maria's sense of how she wanted to behave in conversation with me or with her group-mates. Readers may be feeling what I felt at the time: that there were hints about Maria's sense of herself as a student and her sense of how testing might interact with that but she made no one clear statement to illuminate this concern.

Students often view tests as threats. That makes sense, considering that their experiences are of tests designed to identify the elements of prior study which they do not know completely (Theobald & Mills, 1995). What must a teacher do? Is this an assessment design issue? Perhaps it is possible to design tests which are less threatening--tests which notice what students do know, and notice how they come to understand, rather than what they do not know. However, that might mean having to challenge and reorient the students' sense of the structure of school. "Here we see the imbalance in the teacher/pupil relationship necessitated by the teacher's role in developing the pupils' mathematical meaning. This is where the power, authority, and control invested in the teacher by the institution can be used positively to allow negotiation, interpretation, and meaning to develop, or negatively to impose knowledge in a meaningless way" (Bishop & Goffree, 1986, p. 347). Can tests be designed to be less threatening, if the contract of the classroom still includes that the teacher will judge students' performance competitively? In the previous passage, Maria probed about this connection when I suggested that I had a non-traditional orientation to my testing purposes.

In the above transcript, you were able to see negotiating about assessment, the teacher hinting that the test will somehow have to align itself with the instructional style, at least in terms of its group-centred nature. There is a clear downside to using an innovative testing approach on top of an innovative learning approach. If a student is uncomfortable with the learning approach, a similarly aligned testing approach will increase that discomfort. However, imagine that a student is relatively comfortable with an innovative instructional process, but is maintaining a traditional sense of the goals of the program overall. An innovative learning process becomes even more distinct from traditional programs if it has compatibly innovative testing processes, and so the testing distances the students' experiences even further from what they are used to and from what they may encounter in further learning. Hilde brings up this matter in the transcript below, which picks up where we left this conversation.

Hl: Yeah but then what happens when you get on, and the teacher says you can't work in groups, then what?
MrM: You mean,

Mr. You know why?

MrM: You were addicted to one style of learning,

Sv: Right.

MrM: And now you're going to be addicted to this style of learning,

Sv: Right.

MrM: And when you switch over, you'll go through it all again.

Sv: Right! [laughs]

MrM: If you're good at putting it together.

Sv: Of course we're not getting anything done, so,

MrM: We're not doing any algebra right now, but we are learning about learning right now. At least Maria is, right?

Mo: I'm learning.

Hl: I'm learning.

Sv: Me too.

MrM: Three out of four, anyway.

Mr. All right, I'm learning. [laughter]

MrM: Four out of four so we're not wasting time.

Mr. All right.

MrM: But yeah, don't get addicted to this style of learning, okay. Don't do it five days a week. No don't get addicted to this either because I can't guarantee you anything for grade ten or eleven.

Sv: Okay, I understand.

MrM: Don't just do what I say. I noticed you don't just do what I say, when I read what you wrote down yesterday. Good. Basically what I did yesterday is I deliberately went too fast because I wanted to put lots of important stuff on the table for you guys to play with as a group.

Hl: I got it all down.

MrM: Yeah. And do you understand it all?

Hl: Yeah! [laughs]

MrM: Oh dear! [laughs] Then, don't listen to her, or she'll just teach it all to you,

Mr. She's a good teacher though!

Hl: I'm going to be a teacher, too! [laughs]

MrM: Are ya? Anyway, I'm out of your face. You've got Rectangles, Painted Cubes, and the old stuff to do, and you just asked me if you should do six rectangles or nine.

Hl: I think we should do three. [laughs]

MrM: Yesterday I told you that you would have yesterday and another day to wrap this up.
Mr: I think you should give us more time. [laughs]
Sv: A week! [laughs]
MrM: I think this is good stuff. I'd like to give you all the time you can.
Hl: Yeah, okay.
MrM: But there's limits.

On one level, this conversation was about the comfort level of students with regard to assessment. Despite my expressed confidence in her learning and the learning of the others, Maria has not had her fears and concerns addressed in a way that would allow her to dismiss them. Indeed they are valid concerns. However, she could take comfort that her teacher took her concerns seriously, as I did in making a commitment to find a test structure that did not isolate group members from each other. In turn, the teacher has been able to express his priorities for them, in this case validating that talk about marks and learning is learning about learning, and that's important too. Troubling aspects about the upcoming test remain unanswered (and Maria's sense of a test being right-or-wrong and therefore not matching inquiry-oriented learning will reappear), but it seemed less ominous for all concerned, even as the complexity of the issue became apparent.

It is the process of discourse, the idea of negotiating classroom procedure with students, that is the viable answer to questions such as how to assess innovative inquiry-based curriculum. "It is through assessment that we communicate most clearly to students those activities and learning outcomes we value" (Clarke, Clarke, & Lovitt, 1990, p. 128). In other words, it is not the particular tests that were used which are being recommended here, but the process by which those tests evolved, in conjunction with and in communication with the students. This claim redirects attention from the ultimate choices which were made in this classroom in regard to assessment to the processes that brought the choices forward. This thinking is in line with the context-specificity and learner-specificity discussed in Chapter 3: effective testing for inquiry-based curriculum is not a context-independent instructional decision, in that it is directly tied to deep concerns of particular students, and directly tied to elements of the particular classroom context.

This defined the most challenging element of my personal search for effective pedagogy: How could I assess in meaningful ways, given that such assessment was essential to the ongoing development of the students' approaches to learning, given that the students' history with mathematics assessment provided a pre-defined format which was not only of limited value for my purposes, but made the students generally feel threatened? If I chose to make the assessment program as radically different from what the students expected as the inquiry-style instruction differed from traditional delivery, it could match the instructional program more closely, and could pursue the goal of providing meaningful feedback concerning the students' evolution as learners. Yet, the more radically different I made the assessment, the more I was likely to incite the students' vulnerability (Vincent & Wilson, 1996). Over the course of the program, assessment decisions and issues shaped my own pedagogic inquiry, as when, in the chapter about Martin and his sense of what marks were for, I made the inquiry as public as I could (Chapter 6). As a result, and because marks were a significant concern for the students as well, assessment became a part of much of my discourse with students.

Assessment had become the most problematic element in my personal inquiry as a teacher during the research, and it became a dominant topic of much of the students' interpersonal discourse with me. Yet it was not a particular assessment decision which made the difference for the students in terms of the dissipation of their assessment anxieties. What ultimately mattered most as the students searched for a sense of tentative comfort regarding the assessment we were doing was not the tentative answers (the tests and how to mark them, the report card letters, the role of the research-based instruction in their overall math mark) at which I (and Mrs. Larkin) arrived. It was the constant availability of my thinking about the problematic, and my inclusion of their thinking in my inquiry processes that allowed them to feel comfortable. Surely someone who listened so carefully to them would not ambush them with an unfair test, and if a test or any other instructional decision did cause problems, surely someone who listened so closely would not abandon them to suffer the consequences without repair. Furthermore, the students were able to engage in talk (indeed, engage in inquiry) about what really mattered to them in regard to mathematics and learning and schooling. During my open inquiry regarding how to provide test-style assessment, the students were able to learn about their own sense of vulnerability to tests and assessment, and were able to consider changing their stance in relation to the vulnerability that tests created.

I think this is a generalizable possibility of significant value in today's curriculum turmoil: as teachers deal openly in the classroom context with the problematic of assessment, the students will be able to engage with that problematic for themselves. In so doing, their sense of what learning and schooling is about for them can develop more complexity. As a result, students will be able to construct new orientations, new senses of themselves as students and as learners, which allow an amelioration of the vulnerability and dependence which traditional tests tend to enforce. Perhaps the achievement of the teacher's original goal in this professional inquiry, finding compatible assessment procedures, can be considered a secondary outcome in relation to the student learning which the inquiry and its mutual pursuit can trigger.

This is an ambitious claim, and must be developed through multiple elements of the data which this study generated. In so doing, this consideration of tests and assessment must remain linked closely to the chapter's main goal, the exploration of ontological learning. It is time to switch focus, to the private words of two other students, Lorna and Rob, who (compared to Maria) leave a much less equivocal sense of their views regarding the connection between assessment and their confidence as learners/students. The chapter will then look further at Maria's sense of herself as a learner, when these two students offer us a basis for comparison.

A quiet student learns to value her learning.

> [In] an inquiry-based environment, both teachers and students work toward their own growth in understanding. This classroom environment can promote success for all students. All students explore the mathematical ideas to the degree that reflects their interests and excitement and to the degree that explorations become relevant and important (d'Ambrosio, 1995, p. 772).
>
> The intended modern school curriculum, which is designed to produce self-motivated, active learners, is seriously undermined by classroom management policies that encourage, if not demand, simple obedience. We advocate that a curriculum that seeks to promote problem solving and meaningful learning must be aligned with an authoritative management system that increasingly allows students to operate as self-regulated and risk-taking learners (McCaslin & Good, 1992, p. 4).

Lorna was the group-mate of Benazhir and Rose in Chapter 1. As she showed in her approach to the algebra tiles sheets, Lorna felt comfortable with the program. Despite her quiet nature in whole-class sessions, Lorna often expressed herself thoughtfully in the homework system of written reflective discourse (Chapter 6). Two themes may be elaborated through viewing Lorna's writing. Along with the responses I wrote, students' writing was another frame in which assessment processes were discusses and negotiated. Concurrently, students' sense of personal purpose evolved and can be seen to have evolved through their writing. In this first sample of her writing, Lorna expresses her satisfaction with the homework writing process and suggests that it is effective for her on a variety of levels.

> Lorna: January 27--*Homework: What have the homework questions been designed to help you with?*
>
> They help us understand exactly what we are learning. We can ask any specific question in the homework section and you will help clarify it for us. Also, to see if we can piece together what you are hinting at sometimes. It also helps us review some concepts that we learn. If you can write down what a concept means in words, you most likely have mastered the concept and understand exactly what you are doing with it.
>
> *Sometimes writing down a concept develops the mastery you mention, Lorna. The connection between a concept and "what you are doing with it" is important to a full understanding, and it is a difficult connection to perceive. To tell you the truth, usually when I am hinting at something, it is because it is something in* *your* *mind , and I cannot be sure of all the pieces. That is why I must leave it to you to piece it together! I am glad the questions help! Hey, thanks for saying my answers to your specific questions are helpful. It is nice to get feedback like that!*
>
> *I feel honored (and it helps me think!) whenever I get to read what you think!*

Lorna expresses herself like someone who has clearly distinguished between the two goals of achieving marks and learning. In the above sample she has focused on the cognitive effects of written reflection, showing herself to be thoughtfully conscious of her own learning and suggesting that she deliberately does things to achieve understanding. Marks or any other aspects of the hierarchical aspect of the teacher-student relationship make no appearance here. Lorna is clearly an intentional learner (Bereiter, 1990). Lorna did not feel this way early in the program. In the next excerpt from her homework, written very early in the school year, Lorna expresses what most of the students felt. A second excerpt follows, however, where Lorna expresses her expanding sense of her own learning processes, not only in regard to the math (patterns), but in terms of learning to learn more effectively.

Lorna: September 18--*Homework: Tell about mark worries from Mr. Mason*

I don't know if I am worried. I think I am more concerned. Teachers are always saying grade nine is the most important year in school. You should get the best marks possible, so you can choose the courses you want in high school. I just hope this course, doing both grade 9 and 10, won't affect my marks very badly. Math is my best subject, and my marks in it sometimes make up for other classes. You tell us this will help us and I hope you are right, but so far I'm not very convinced.

I feel honored by the blunt honesty in your last sentence, Lorna. I am glad you are not automatically believing what I say. Do you believe this:

You will probably find yourself able to take anything you wish in grade 10. In reality, you will probably have the marks to select any course. The challenge in grade nine is developing COMPLEX learning skills to deal with the COMPLEX content in those courses. People who get good marks in grade 9 by listening, memorizing, and practicing are NOT prepared to engage complex topics, even though their marks are good. You can have BOTH: good marks and good preparation!

Lorna: October 20--*Homework: What mark would you give yourself for the first three areas? Why?*

I would give myself a good mark (I don't know exactly what numbers) because I tried really hard and worked on these sheets. Some questions were quite difficult, but I didn't leave them blank. I learned a lot about different patterns and different ways to use them. I think it will help my math by doing these things. Being confused and not understanding questions make me think and try even harder to get the answer. I believe that when you make us think like this for so long and keep giving us new questions each day really gets our brains moving.

Thanks for thinking about this, Lorna.

I am glad you believe in what we are doing together, Lorna. Also, it pleases me that you understand that you were learning when you were confused. You seem to think that your mark should have something to do with your effort when things were difficult, and also the amount you learned (about patterns and different ways you could use them). That's about how I see it, too. I agree that it is difficult to decide "exactly what numbers" would measure such important stuff.

Lorna perceived her own learning processes in terms of the knowledge and general skills that accrued to her as a result of those processes. At the same time, she was very conscious of the structures of school in which she was active and successful. The next homework statement elaborates on my response above. Please notice how it expresses a different view from the one from September 18 about the relationship of current schooling to future schooling: in the beginning of the year she expressed that good marks are what allows future success, but now she makes a different claim.

> Lorna: November 5--*Homework: Answer my response to your marks comment.*
>
> I agree with what you said. I think that after we finish this course, we will have learned several different concepts about patterns and algebra etc. I am glad that you agree with me about what our marks are, but grades aren't as important as learning and understanding. If you don't understand something you will never make it through high school and university. It's not the answers that are most important here, it's how you get these answers that really matters.
>
> *This is a very big idea you have said, Lorna. I hope that when you care about "learning and understanding", the marks work out for you. What happens if you have to choose between understanding and marks? You have answered that, and bravely.*
>
> *You're right about answers not mattering. Who cares, really, about mini-cubes having paint on them? You're right about your method mattering. If you can figure this stuff out, then you are ready to do first-quality learning. Yet, if this is true, why are tests mostly about answers?*

What does it mean that Lorna shifted her stance regarding what she valued in math class from the marks she earned to the understanding she constructed? In one sense, it means that she was less amenable to the control structures of teachers: teachers often use marks and tests to communicate to students what they expect students to do (Peirce & Stein, 1995) or to manage their behavior directly (Stiggins, 1995), and Lorna would accept that direction readily only to the extent that the marks and test mechanisms align with the learning and understanding she desired. At the same time, however, she was clearly more open to the influence of a teacher who would discuss her priorities with her. Perhaps more significantly, in the minute-by-minute functioning of the classroom Lorna was pursuing something from a stance that viewed herself as having authority. For instance, when doing the algebra tiles sheets (Appendix B), she had a way of deciding how much time to spend on the open questions or in conversation with group-mates about the questions, without demanding delineation by the teacher. Rather than depending on the teacher's instructions or assignment of marks to guide her use of time, she could choose for herself based on her estimation of the value of a particular behavior in contributing to her goal of understanding. Although she depended on the teacher for structuring the opportunity to learn, she was relatively independent of the teacher in managing her efforts within that opportunity, intellectually autonomous in that regard (Yackel & Cobb, 1996).

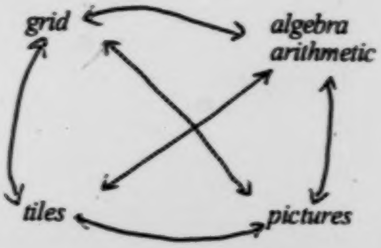
This does not mean that marks ceased to matter, as the next sample of Lorna's homework shows. In the writing of the second letter to parents to accompany their report card, I indicated for each group a sentence such as, "If I imagined 100 groups of grade nine students, this work suggests to me that Lorna's group would be in the top fifth in their learning processes." This percentile-like comment attracted some students' attention: it sounded like a mark, but it wasn't the same as a traditional mark. Lorna inquired about the difference, and my answer turned down her request for a traditional mark. However, she identifies a particular purpose when she justifies her request for a mark.

> Lorna: January 28--*Homework: Prepare an idea for me to respond to:* *marking.*
>
> Are you going to be giving us a percentage mark when we finish these algebra tiles? I think I would like a percentage so I know if I am really learning something or not. It helps give me a clearer picture of my learning.
>
> *If you want a % mark, what will it come from? Would you like one out of my head, or would you like a test or a project, or for me to give you scores on homework, or what? You decide, and I'll do it.*
>
> *I have 5 sheets for you to do with the algebra tiles. Maybe they will help you know if you're "really learning something or not". Regardless, I will always give you my opinion if you ask for it. You deserve all my support.*

As Lorna learned by doing the algebra tiles sheets, she apparently accepted my invitation to judge her learning for herself. I suspect that her request for a percentage mark was less a request for a mark with more accuracy built in as for a mark which belonged only to her rather than to the group. In the following homework response, which actually comes from the day on which Chapter 1 is based, we can see Lorna deciding for herself that she is learning competently.

> Lorna: February 3--*Homework: Respond to any one idea from last week.*
>
> I asked if you were going to be giving us a percentage mark for what we are

> taking. I agree with what you said. I hope these five sheets will tell me if I am really learning something. From what we did today (the first sheet), I think I am starting to understand these algebra tiles. Working with the pictures and tiles is giving me a clearer picture of algebra, and how to do the equations from it.
>
> *Confidence is very important to successful students. In today's writing, I hear your growing confidence. The exciting part about that is that you have found some confidence BEFORE you are sure you understand. You are, as you say, just "starting to understand", but you are already noticing your progress, and using it to feel your on-going confidence! Great!*



> *I am also glad to hear you connecting the tiles, the pictures, and the algebra arithmetic. Today, I asked you to include the grid multiplying in that package. That's a lot of connections!*

Two things were evolving. Lorna was changing her sense of priority, choosing to view her activity in math class as being that of a deliberate learner rather than a student doing assigned tasks. In one sense, this was about Lorna developing a more autonomous relationship with the teacher, mutually negotiating a different role structure. However, independent of the teacher or the subject area or the school situation in which this change occurred, Lorna was making a fundamental shift in her sense of herself: she declared herself to be a learner more than a student. The teacher was still a significant relationship, but instead of the teacher defining that relationship in terms of the authority structures of the school, it took shape as the student and teacher determined how to learn and foster learning. Lorna's relative autonomy in regard to her sense of values regarding marks allowed an interdependence with the teacher in regard to her learning (Chickering and Reisser, 1993). This was the second element which was changing: more than desiring marks as measures of her student performance, Lorna desired assessment feedback to confirm and guide her ongoing learning.

Compared to Maria, Lorna was less oriented to influencing any pedagogic practices which I might have established, including assessment practices. Instead, she anticipated establishing a meaningful and successful student stance in relation to the instruction through her participation in the homework discourse. Maria too was proactively negotiating her relationship with the teacher while discussing assessment concerns. Unlike Lorna, however, Maria accepted my invitation to consider my pedagogic practices as negotiable, and was willing to participate in a more mutual evolution of effective stances, student with teacher-researcher. In a traditional sense of the teacher-student relationship, we would find Lorna's stance much more appropriate, in that Lorna committed to adapting to my choices without judging those choices. On the other hand it may be that Maria was willing to pursue a more mutually satisfactory relationship than would be possible if one person took the sole burden for adapting to the other. The stances were very different, but as we juxtapose the two students' stances it would be more appropriate to attempt to appreciate their differences, not just decide which is superior (Bateson, 1989, 1994).

In the passages which opened this chapter, Maria expressed her vulnerability by making direct reference to the mark which the test would generate. Yet it would be too easy to assume that it was primarily the marks themselves that mattered to her. Maria's references to the mark could as easily have been a way to express concern about her performance on the test as a meaningful indicator of how well she had been learning, so her concern could conceivably have been as learning-oriented as is Lorna's. Alternatively, Maria could have been concerned primarily for the indirect impact which poor marks could have. For example, Maria might have been worried that a low test mark might affect her full membership in her learning group. These options are not possibilities to be pursued here, but are posed only to serve as examples of possibilities. They suggest that it would be inappropriate to use the comparison to Lorna's more independent and more purely purposeful orientation to simply categorize Maria's orientation as less desirable and therefore less developed. It could well be so, but it could also be that for Maria issues of assessment are more deeply implicated in various elements of Maria's sense of herself than for Lorna. Maria learned in a more interactive way than did Lorna: as our experiences with each of them in previous chapters indicate, Maria's involvement with her group-mates was more significant both to her learning and to theirs than in Lorna's case. Maria also learned in a more interdependent way: her group-mates' openness to Maria's influence (and my openness to her influence) was entwined with Maria's vulnerability to the opinions others expressed of her (and is a teacher's mark anything else?).

An independent student learns interdependence.

> To develop their own knowledge, students must be able to perceive themselves as autonomous. To engage in complex knowing, they must be able to move beyond autonomy to interdependence (Baxter Magolda, 1992, p. 363).

Listening to a third student at this time may sponsor an appreciation of differences among the students rather than better-worse two-way comparisons (Bateson, 1989, 1994). Rob also expressed his concerns about the upcoming test, but did so from a completely different sense of his role as a student. Samples of Rob's homework will illuminate his sense of his identity in the classroom and the changes he chose to make, but they can also be relevant in thinking about Maria's statements in the chapter's opening conversation.

This is Rob's first significant appearance. Also a competent student, Rob tended to view his job as a good student as cheerfully doing only what he had to, letting his intelligence carry him to above average achievement. Rob knew what tasks were essential in school, because those were the tasks to which teachers assigned marks. In a sense, this stance allowed Rob to be relatively independent of the teacher's influence: he did not need to do everything he was told, but by doing the essentials (and doing them cheerfully) he seldom caused teachers to assert their authority. In the inquiry-based curriculum, Rob's stance allowed him to do well but not excellently in making sense of the patterns activities: his use of his natural intelligence to make sense of mathematics without extensive effort was well suited to group investigation, but his reliance on a teacher's prioritizing through the marking structure left him without a sense of what should be his specific focus during the inquiry process. Without an aggressive marking structure outlining the details that needed their attention, Rob's group sometimes was less than rigorous in investigating those details.

Rob was not naturally reflective, and coming into the inquiry-based instruction he had no sense of writing to a teacher other than doing what was required for marks to satisfy the teacher. Similar to Ivan (Chapter 6), he tended to write little or nothing for the homework for the first half of the program, after having made sure that I meant what I said that it would not count directly for marks. In fact, his longest writing during the first months of the course was a tongue-in-cheek apology for the paucity of his responses. My reply once again suggests other than mark-oriented (teacher-oriented) reasons for doing homework.

> Rob: November 5--*Homework: Answer my response to your marks comment.*
>
> My reply to your reply about my reply can't be replied because the first reply was not replied so I can't reply. Sorry about not replying. I am behind on many replies. Please reply to this reply.
>
> *This homework was your chance to notice how you are learning, Rob, so you can use the skills in other situations without me. You're the one who isn't benefiting from our written conversations. It's your replies that matter.*

Shortly after this, Rob started to provide thoughtful answers to some of the homework questions. It is unlikely to be just my replies that triggered this; nor was it likely just my mentioning Rob's approach to his homework in the letter to parents which accompanied the first report card. Rob decided he'd better try out a process which others were saying was good stuff. As a consequence, when Rob found himself worried about the test which we had talked about in class, he was able to write what he thought. In his writing can be recognized Rob's assumptions about the role of a test and the responsibilities of a teacher and student to prepare for a test. Also apparent is how far apart his conception of algebra and mine were. I replied to each of five parts of his response, and inserted numbers into his response to correspond to each part of my reply. As I read my reply now, I wonder whether I might have overestimated Rob's spunkiness in my decision to reply with spunk myself.

> Rob: December 13--*Homework: What should the pre-formal algebra test look like?*
>
> I think that the exams themselves are not a good idea. I personally feel that I am not perfectly okay with all of the things that we've been doing (especially the first 2 months or so, e.g. Handshakes, Passways and Odd Jobs). I would rather have us to spend a few classes and you tell us how this all goes together (Odd Jobs, Handshakes, etc.). *[5]* I also know that many of the other students feel the way I do. But don't get me wrong, I wouldn't mind getting a test *[3]* to see how well I grasped the ideas so far, but I would feel better about myself *[4]* if I first understood initially then have exams on what I know.
>
> I know that I have not exactly answered the question, but your algebra and the grade 9 textbook algebra is ... well different some way. *[2]* I DON'T KNOW. But if you want an example of an algebra question I guess it should include things

like:

1. $\dfrac{a+3}{-(b)^2+1} = [\_]$ Evaluate and please give integers for *a* and *b*. RIGHT??? *[1]*

1. What if I asked you that question, Rob, but I didn't ask you to EVALUATE it, but I asked you to explain what it MEANS or what is it FOR? Do you really want a test with questions with NO MEANING and NO USEFULNESS? Why? (By the way, the question CAN have a meaning and a use, and I hope you learn those, as well as how to "evaluate" it, when the time comes. Be patient, but insist on understanding.)

2. You're right. The textbook is designed to keep you busy and it is designed as part of a program to eliminate 2/3 of the students and it is designed to keep you passive and to ignore your personal complexity. My program isn't. I think you did a GREAT job of answering the question --such spunk, such insight, such concern, such forthright expression, is BANG ON!

3. Yes, one reason for tests is to help you see how you are doing. It also forces the teacher to show you exactly what is expected of you, and that can help, can't it.

4. You deserve to feel good about yourself as a student, Rob. I know you already do, because you take risks and you try hard stuff. I agree that exams can change that if you don't know what it asks for.

5. This is when I feel sad. It's not your fault, but you have been made to believe that the most surefire way to learn is to be told by the teacher. You have been made to mistrust tests because they are trying to trap you in your weak area. I think it is WISE to mistrust tests! But no way am I going to waste your time with:

--simply telling you a bunch of simple facts

--giving you a test for you to say the simple facts back to me

--ignoring the special learning style that you have, and that you have developed with your group and with me.

But -- how do we get you a fair and accurate mark for all this? Maybe that's just not a good idea in the first place.

P. S. This is a spunky reply, to honor your spunkiness! You may challenge ANYTHING I said here!

What does this mean, within the framework of this chapter? On one level, it shows the power of the homework as discourse. For the teacher (and in retrospect for the researcher), the discourse provided a window onto individual students' thinking and concerns. It also provided the students with a window on the teacher's thinking and concerns, and between the two lay the possibility of negotiating a mutually satisfactory relationship. The vitality of this style of discourse was not so much dependent on its information flow potential or its reflective awareness potential as on its authenticity, its potential to influence the behavior or choices of the person with whom each was communicating. In this example, Rob believed that his writing could conceivably make a difference to the decisions I made about a test which concerned Rob considerably. As the teacher, I believed that asking the homework question for Rob to consider and providing a response for Rob to read might help Rob to find a sense of purpose with regard to the homework generally and help him develop an appetite for deliberate learning rather than doing the minimum required. More generally, both of us believed that their participation could improve the pedagogic relationship between the two. When discourse participants believe that their participation can change the behavior of other participants, the discourse is authentic, rather than just an exchange of information (Freire, 1970). "We can come to understand their perspective only if we create a classroom environment in which the students' voices are heard, in which the students' interests are explored, and in which the students are called on to give direction to the classroom activities" (d'Ambrosio, 1995, p. 770).

On a broader level, however, for us as retrospective readers, Rob's homework brings to light the web of interconnectedness among content, motivation, students' sense of what counts as knowledge, and students' sense of self in relation to the school system and the teacher. More directly, Rob's response suggests valid misgivings about the misalignment of traditional tests (the only kind of tests which Rob can imagine) with nontraditional curriculum. If students perceive the act of writing a test as the retelling of details that have been told to them, then how can they feel prepared when there has been very little telling? Even when the student recognizes the learning value of non-delivery instructional styles, the test (if perceived in a traditional sense) is threatening. It is this which Rob was more willing to express directly to me than Maria was. In comparison to the chapter's opening transcript, has Rob said what Maria meant with her statements about a test being right or wrong? The concept of *test*, limited by the prior experiences of these students, implied to them a dangerous mismatch with the kind of preparedness which the innovative learning processes had developed. "The discourse is shaped by fundamental values about knowledge and authority--about what makes an answer right and what counts as legitimate mathematical activity and reasoning" (Ball & Schroeder, 1992, p. 68).

It may be useful to think of the students' sense of curriculum and testing within the metaphor of territory. In a traditional sense, the teacher guides the students across new terrain, emphasizing the features of the landscape that the students are to know well. The tour of the terrain stops on occasion, so that students can get more familiar with landmarks which the teacher has flagged as important, and stops on other occasions so that the students can repeatedly traverse the ground which connects two landmarks until a trail is visible. Before the test, the teacher ensures that the students are aware of what features of the territory will be emphasized, and the students anticipate repeating processes they have practiced before. With inquiry-based curriculum, students' exploration is more significant than the teacher's role as guide in terms of what features will be significant. The students are responsible for confirming their sense of the landmarks they encounter, either by recognizing for themselves their significance within the lay of the land or by checking with the teacher. As you have seen in all the chapters, in general these students had adapted successfully to this approach to learning. However, neither their experiences with tests in more traditional instructional settings nor their inquiry experiences in math class provided them with a sense that tests as they understood them could align with the exploring and map-making they had been doing. It is from this dissonance regarding assessment and the resulting negotiation about the nature of the test that our opportunity arose to perceive and understand Rob's and Maria's senses of themselves as students and as learners.

What happened subsequently to Rob? Although he continued to forget to do his homework writing at times, Rob began to word-process thoughtful answers to each day's question on his computer. Also, Rob upgraded his intensity with regard to the in-class learning activities, within the limits which his learning trio allowed. The next homework sample below provides a view of Rob at his most thoughtful: along with a statement of the newfound value he placed in reflective writing, he expressed his new epistemological awareness of the cognitive result of expressing questions and answers in many ways. I have followed this with a second sample where Rob suggested a significant change in his sense of priorities regarding marks. From having perceiving marks as a part of a system that informs the student what is essential work, Rob expressed a view of marks as valuable feedback on how he is learning.

> Rob: January 27--*Homework: What have the homework questions been designed to help you with?*
>
> I feel that by being asked those questions I know more about what I want and have a deeper view of the subject. The questions really made me think in a

different way. I mean that I have never been asked those kind of questions nor have I asked myself those kinds of questions. What I mean by "kinds" is that they're in a completely new perspective. I did find them difficult at times, or I never really understood them, but tried to answer them to the way I saw them. By the way, I am really enjoying the Diagonals and Radicals Discovery sheets. It really does prove that there are many ways of answering and describing a question or answer.

I'm glad you're both "enjoying" and benefiting from the Radicals Discovery sheets. I notice you pointed out the value to you of "many ways of answering and describing". Understanding is much more than getting one right answer! I wish we had another class or two with these sheets, but oh well.

I noticed especially your sentence about not having asked yourself these questions. It makes me glad I asked them, and glad I let you answer them your way, when I see how much they have come to matter to you. From now on I will only be here twice a week at most, so feel free to ask yourself a question, even if I don't give you one, and when I arrive I will respond. Responding has made me think in a different way about teaching! Thanks again!

Rob: January 28: Prepare an idea for me to respond to: marking.

I feel that the marking system you have right now is perfect. I am sick and tired of looking at dumb numbers. The numbers mean absolutely nothing, well maybe perhaps how well I did on the assignments and tests. Otherwise that 80 or 85 doesn't mean I sincerely KNOW the material. I prefer a comment from you like you are doing right now because you tell me and my parents more of what is important. They most likely will ask, "But where is the mark?" only because they have been raised and taught the old way, just like we are in other classes. I am all for the marking system you have designed, really love it!!!

Glad to hear it, Rob. Your parents are right, however, to expect a traditional description of how your knowledge compares to your classmates. My marks go along with Mrs. Larkin's, to create a package. We hope to be able to communicate in ways that help you continue to learn, and that help your parents continue to help you.

By the way we designed this marking system as a team, the 21 of us. Our communication is central to my opportunity to make choices that please you and the class. Maybe your parents could be part of the team somehow too.

In Rob's response is an important reminder to me that marks and other evaluative descriptions of a teacher's opinion of a student's performance are not just a communicative act between a teacher and a student. They are significant to the student in their relationships with others such as peers and (in this case in particular) parents (F. Smith, 1995). This means that the re-negotiating of the meaning of marks within the teacher-student dyad is constrained by others' interpretations of marks and their purposes. It also means that marks affect not only how the student perceives himself with the teacher serving as mirror, but they also affect how the student will be perceived by others who also help to shape his self-image. Marks are ontological within the school context of the teacher-student dyad, but they have ontological ripple effects beyond the pond into which they are dropped. This theme manifested itself in all the students' discourse about marking, and will be prominent in each of the two student-centred chapters to come. In consideration of Maria, this may be a further reason not to hold Maria's expressed concern about marks as indicative of an institutionally instrumental view of tests and marks: Maria's marks may have been fundamental to her parents' sense of her success as a student, regardless of Maria's own more independent sense. It may be wisest when students discuss their concerns about marks to hold as an open and probably complex question what their statements indicate.

Does wondering about independence and dependence help us understand Lorna, Rob, and Maria? Lorna exercised a sense of independence implied in her preparedness to adjust to whatever the teacher chooses, while holding her own opinion of her learning as valid. Rob independently decided with how much effort he would respond to tasks his teacher set, but he depended on their relevance to assessment to make that decision. In the beginning, that left him especially vulnerable where tests were concerned, yet his distance from the teacher in terms of unassessed communication channels left him relatively unable to influence the assessment choices that the teacher made. In this vein, Lorna's lack of open participation in classroom discourse and her extensive participation in the homework discourse provided very different levels of opportunity for her to affect what a teacher chose. Maria, on the other hand, was not only much more interactive in all channels of communication with the teacher: she was interactive in regard to all topics the teacher broached, including the choices the teacher had yet to make. This interdependence opened more vulnerability than Lorna's independent or Rob's arms-length relationship with me, as was apparent in the transcripts which opened this chapter. Yet it also provided greater efficacy in relation to actions of significant importance to her. Her openness to influence provided channels by which she could influence others. This was probably also true of her interdependence with her learning group, as was illustrated in Chapter 5. Attempts to use independence and dependence as a dichotomy would benefit from including interdependence as an alternative, when it comes to interpreting students' interpretation of their relationships with the teacher's authority. Both invulnerability and independence make poor ideals for students and, I would suggest, for teachers as well.

The image of Maria developed in this chapter and Chapter 4 has not had a longitudinal element as yet, but with the homework discourse with Rob spanning a significant month, we could justifiably ask how much Rob's stance with assessment changed. Rob's statements suggest that he at least changed from significant discomfort to (apparent) complete agreement. Significantly, Rob's statement that the system was "perfect" was written after Rob had experienced a couple of tests within the program, so fear of the unknown would not have been the same factor it was in the earlier statement. It is conceivable that Rob did not change at all--perhaps he had just determined that the assessment system was harmless. However, even in the letter to parents which Rob's comment addresses, there were some pointed critical comments. That Rob was indeed stating sincere appreciation for the feedback was supported by his increasingly rigorous approach to the program, something evidenced in his much more thoughtful homework response from January 26. I believe that Rob's sense of having been a part of the evolution of the program, its assessment especially, helped him develop this sense of respect for the assessment whose guidance he accepted. Rob changed his approach to being a student of mathematics, seeing himself differently within that role compared to how he saw himself at the beginning of the inquiry-based instruction. "When a practitioner [or student] becomes aware of his frames, he also becomes aware of the possibility of reframing the reality of his practice" (Schon, 1983, p. 310). Still, the absolute nature of Rob's feeling regarding the "perfect" assessment system bothered me: given the complexity of assessment and its central relevance to the life of a grade nine mathematics student, and assuming that ongoing inquiry into one's own relationships with school and with learning is a good thing, at least a little dissonance in that regard is probably better than total comfort.

Lorna and Rob were students with remarkably contrasting entry orientations to the work of learning mathematics, and correspondingly contrasting senses of the role of marks and the tests that generated them. Yet, both engaged in actively negotiating with me what tests and marks could be, and both found comfort in the processes that came to be. Lorna confidently prioritized learning in her decisions as distinct from the earning of marks. She heeded her own sense of her success in learning, appreciated meaningful discourse in relation to her learning, and respected a teacher's marks to the extent that they inform her about her learning. Rob, on the other hand, was not as confident. Marks had the power to make a difference to his sense of himself, directly (as they interacted with Rob's sense that he was quick if undisciplined in his grasp of concepts) and indirectly (as they affected his view of himself through their influence of the views which others had of him). However, he learned to value descriptive feedback about his learning, and incorporated some of that advice into inspecting and adjusting his stance as a student. He recognized his competence to think about his learning and he recognized at least to some extent the potential value of thinking about learning. For both students, conversation provided a framework not only for expressing their views, but also for affecting their context and for adjusting their viewpoint. As they adopted more complex and more suitable viewpoints about their learning processes and their stances as students, they were learning ontologically. "Knowledge about social life is not to be viewed as a 'reflection' of what there is, but as a 'transformation' of experience into a linguistic ontology" (Gergen, 1982, p. 202).

The homework comments and responses document the epistemological and ontological learning that took place during the mathematics inquiry. When ontological concerns are expressed within a private written exchange over time, it is possible to track (and to influence) changes in students' ontological stance. Ontological issues are deeply personal, and the homework structure described here is compatible with the expression and development of deeply personal issues, especially when a topic like assessment links the two participants. However, students did not limit to private written structures their discourse about assessment and their roles as students and learners. Much of the reorienting which the homework describes took place during mathematical activity and whole-class discussions. To view the reorienting in progress, the focus of attention will be shifted now, first to a whole-class teacher-led discussion, and then to a small-group discussion, before closing with Maria's voice.

The whole class talks about learning, teaching, and studenting

> The discourse of student experience supports a view of pedagogy and empowerment that allows students to draw upon their own experiences and cultural resources and that also enables them to play a self-consciously active role as producers of knowledge within the teaching and learning process (Giroux, 1989, p. 148).

The tension between in-school assessment structures' one-size-fits-all nature and the distinctly individual nature of students' ontologies can provide significant fuel for whole-class conversation. The class of December 13 followed immediately after the Rectangles Families activities of Chapters 4 and 5 and just before the patterns tests described in this chapter. At the time I considered it the most significant of all the classes in the inquiry program, although it was a class in which no one did any mathematics. Its main focus was the students' concerns about assessment within the inquiry program and its possible effects on their mathematics marks. Yet it was also about each participant's epistemology and ontology on another level. On one level the primary outcomes of the day were that the students felt better about their role in the inquiry program and we made some general plans for the assessment that was to follow. On another level, the value of the class was in the conversation itself. Students expressed concerns about what counts as learning, and what should count. They expressed concerns about what counts as studenting, and what should count. They did so because they perceived the opportunity to help shape the instruction of the program (specifically, the test) to better match their views, and within the authenticity of that interaction they opened themselves to shaping themselves to better match the program.

In the transcript below, I open the conversation commenting on the discomfort which some of the students felt with the inquiry sequence dealing with the Rectangle Families (Chapter 5). Those students who were uncomfortable had been operating with different images of how they should be able to learn math than I had as I led the inquiry. With me accepting responsibility for the mismatch in operating imagery, the students offer a variety of diagnoses and suggestions, and I respond to each. My answers always begin by acknowledging the student's key statement.

In the students' expressions are evidence of different stances on what counts as knowing and how learning takes place, similar to what was portrayed in Chapter 5 within the small-group inquiry. However, there are also expressions which are examples of different stances on what students' roles could and should be, in relation to what math teachers' (my) roles could and should be. This is what we saw in Lorna and Rob's homework interchange, but in a classroom context, the statements have a particular impact on the teacher-student relationship. Throughout the discourse, Maria's involvement is extensive, purposeful, and positive.

MrM: One example, if we back up a couple of times. The last two classes were on

December 1 and December 2. My plan for those classes was, okay, in the first half of the first class, I'll just give you guys a brief overview of the whole package that we've got to go for, for the next little while. I'll give a brief overview, sort of as if I'm telling you okay here's the road you're going to walk, and after a few steps you should see this, and you should see this, and then you take a few more steps and you'll see this and this and this. Now I'm not going to tell you more details than this, because you're going to get there in a few steps. And then I tried to shut it down after half a period and give you guys a period and a half to make that little trip. Well I blew it, because,

Ma: [Martin] We got lost.

Rh: [Rhoda] Yeah, we took the wrong turn. [laughter]

MrM: You guys got lost because you thought, and rightly so, because you thought I was telling you to start walking. So here I am telling you about mileposts you're going to get to when you start walking, and you guys are trying to learn it all on the first go. And good for you, because you're used to learning things on the first go, you're very bright. And you guys got frustrated, and that was one of the times I had my head in the clouds and didn't notice. Now you slowed me down and we spent the whole period talking about that stuff, but I certainly didn't slow down enough, because I was only trying to provide you with an overview. I blew it. But when you're going for the gusto, you've got to blow it sometimes. As long as you fix it up after, and that's where you guys can help. Maria.

Mr: [Maria] I think what's frustrating for me is that I never really understand the concept that we are learning, and I don't understand what it's leading to, and I don't understand if the answers I get are right or wrong,

Hl: [Hilde] Right!

Mr: Because it's not like the book where you say, Okay I've done this, and I can check it over. It's like, we don't know what we're doing and we come up with something and we're still confused and we don't know where that's leading us. I'd like to know more about what we're doing and why we're doing it, more information on it, because when you give it to us we're sitting there thinking, What are we supposed to do?

MrM: You always think about things in a lot of different directions, and in a situation like this, that pays dividends to me. In a math class, the way you think about things in a complex way will always pay you dividends as a learner. You said just a whole pile of things there, right? You mentioned wanting to know if your answers were right or wrong. You want to see your progress as you go along, like you can in a textbook. You feel good when you can say, Yeah, we got the first sixty pages of this sucker put behind us!

Ss: [Various unidentified students] Yeah.

Maria has expressed key ideas concerning inquiry-style learning compared to traditional mathematics practice questions. First, student don't get to hear what is the right procedure and what the answer should be before beginning, so the student must begin without a certain sense of direction and destination. Second, there is not a specific answer that, when it is obtained, lends some confirmation that the student is progressing well. Maria knew that in this style of learning the process (and the resulting understanding) are different: she had significant epistemological awareness of what matters. However, she was also aware that her sense of herself as a successful learner/student was at risk: for her, uncertainty during student activity and the unavailability of right answers to confirm her activity left her unable to feel confident that she was the capable learner/student she wanted to be. Below, a little later in the conversation, I return to the question of instructional style, addressing what I take as epistemological concern, but Maria brings me back, hard, to the question of student comfort. For her, it isn't a matter of acknowledging the difference in epistemology that comes with inquiry, but of finding ways to comfortably perceive herself as the successful student she was during traditional instruction.

MrM: Have you ever noticed that when someone tells you a story that's supposed to mean something to you,

Ro: [Rob] A moral!

MrM: Yeah. You can sleep through the story, right?

Ss: [Various unidentified students] Yeah. Yeah. Sort of.

Iv: [Ivan] Like in church?

MrM: You can sleep through what the priest says, what the minister says, as long as you're there for the last minute. in the last minute, he's going to tell you what the heck he was talking about, right? And I believe that you guys are sort of addicted to that in school. You know, normally, if the teacher's going to talk to you about multiplying integers, he's going to talk all about blue chips and red chips, and blah, blah, blah. But as long as you wake up for the end where, you get down to the point when you write it in your notebook, okay, if the signs are different, it's negative; if the signs are the same, it's positive. Gotcha--why did you go through all that chip stuff anyway? And then you go and pass the test and make yourself think you're right.

MsL: [Mrs. Larkin] Until Mrs. Larkin goes and asks you questions that say, Tell me why?

MrM: Yeah,

Kl: [Kelsey] That's difficult.

MsL: Cause you're used to sleeping through that stuff right?

MrM: Or the next year, when you're in a little grid multiplying, and you have to jump from multiplying two negatives or positives suddenly to adding up two negatives or positives. If you're only going by rules about integers, you can't make the switch quick enough, you've sort of got to have an understanding, right? Back in grade eight when you first did all that chip stuff, you couldn't tell that you were going to need an understanding, instead of just memorizing rules.

Ma: [Martin] In grade eight, we didn't do the chip stuff.

Iv: We used money.

Kl: Yeah, we just kind of were told,

Rb: [Ruby] How it works.

Kl: This is what we do, this is the homework, go and do it, you had the whole class to do it.

Ss: Yeah. Yeah.

Iv: Yeah. That was fun. That was fun.

MrM: And you can always tell you were getting stuff, right?

Kl: He'd go by the book, he'd do every page. He's go like, he'd explain it for five minutes, this is what happened, this is what you do, and then you get the homework.

MrM: And all the content was so simple, that it would fit on one page, right?

Ss: Yeah. Yeah.

MrM: No big issues, every little thing, every little piece would pop in. I know you guys were learning Pythagorean Theorem a little while ago, and that's a great topic for grade nines. It's a beautifully sized chunk. It's not small and piddly like positive plus negative. Positive plus negative is a really tiny little chunk. You've got a bigger picture than that, right? And, painted cube is a huge chunk! When you think of all the algebra that is coming at you because of that, whoo! The Pythagorean Theorem is just a nice chunk, and it's just barely possible to get that to fit on two pages of your textbook, as long as you save the word problems for later. But there's a lot of stuff coming at you, guys, that isn't going to fit on two pages. And that's where somebody's got to help you make the transition from going page by page, to taking on bigger stuff.

Mr: Okay, I just want to know, Where is this course leading us to? What are we supposed to establish at the end? I mean, is there any reason? I mean, okay in grade nine we've learned this and this, right? Can you tell us what we're basically supposed to learn so we can go over it again and concentrate more on what we're supposed to be learning?

Kl: Can we finish what we start?

Hl: [Hilde] Can we have a goal?

When discussing an important issue as a class, it's wonderful when the students hit the nail on the head, but I was feeling somewhat like I was the nail. When students express what they expect of their teacher, it can be quite valuable, but most teacher-student relationships do not have room for that kind of discourse and consequently students might not handle such an opportunity effectively. However, if teachers must achieve access to students' sense of learning and sense of studenting (sense of self), to help them to develop richer understandings more compatible with inquiry-based learning, perhaps they must model that openness. It is in authentic discourse, where the possibility of one's words having an impact on others' choices is genuine, that students are most open to considering significant changes in themselves. In this case, the questions by Maria, Kelsey, and Hilde are sincere requests for help: they don't want me to go on justifying the switch in instructional style, because they are already believers in the principles behind the change. But they do want to understand the goals of the new instruction better than they do. I responded assuming this positive intention, assured them I was on their side, and was rewarded by their further involvement.

MrM: Boy, I'm sure glad I didn't answer you right away because I love the clarifications we got there. Make sure they both got heard. Can we finish what we started? Kelsey said. And I like that, because she's talking about, We. You're not just asking me to do it all, and I know that, Kelsey, you like doing your own work. And the other one was, Can we have a goal? In general I guess, my goals pretty directly are your goals. But a lot of your goals you haven't put in words yet. You want to succeed at grade nine at a really high level, and you want to be prepared for grade ten, either so you can challenge the exam, or so that you can blow away the grade ten math course like you've never done before, if you take it, right? So those are your goals. For me to help you with that, I have to help you understand some very specific content. In this case algebra. And I have to get you guys to make a huge jump in your learning style. Okay?

Mr: So basically, you changed the small things that are really big things, sort of? Like the things that will, it's hard to understand now, but once we have it, we'll go faster in grade ten, eleven, like that?

MrM: You're getting a lot of background experience, for a lot of this stuff. Not knowledge, and not understanding, but just background experience. Now you're also getting some understanding, and you're also getting some knowledge. ...

It was important to me that on this day the students control the flow of the issues. However, talk about learning often swings to talk about instruction, which means that issues of epistemology and ontology are at arm's length for the students. That's fine when wanting to demonstrate openness and willingness to deal personally with issues, but it isn't as likely to be about students and the changes they want to make. On the other hand, talk about assessment is more likely to include statements about what counts as understanding, what it means to learn well, and how students can position themselves in relation to teachers and institutional structures. This is talk much closer to the students themselves. Because of the upcoming but as yet undefined test, the conversation did turn to assessment soon afterward.

Hl: Okay. I just wanted to say something about this test okay? Well now that we have this, and we have a better idea of what is going to be on it, I have no idea how you can give us a test when we don't understand this.

Sv: And everybody learned differently, anyway.

Js: [Josie] That's my question too.

Ss: Yeah.

MrM: Oddly enough, first, whatever is fair should happen, period. So, the only way I could do something that's unfair, is if I've made a mistake about what's fair. The only way I will do something that is unfair to you guys is if I make a mistake guessing what's fair. And then it should change.

Something has to be done to change that.
Js: Instead of guessing what's fair, why don't you ask us?
MrM: Okay. What's fair? [laughter]

The conversation still had me on the hotseat. Josie was not as comfortable with the principles behind the instructional change as the others who have spoken and she reflected her discomfort aggressively. Beyond the aggressiveness, however, students' sense of needing confirmation of their understanding before entering a test was well expressed, as was their recognition that their learning during small-group inquiry was relatively personal, even idiosyncratic. On a general and positive note, there is direct linkage in those two student comments between the test and the students' learning or understanding. This theme is repeated in the next portion of transcript. The students suggest that tests as they understand them demand a standardizing of instruction and learning if they are to be fair, something that's not a quality of their preparation. I reply by framing ideal assessment as supportive for good learning and as sensitive to individual differences. This is similar to Maria's concerns which opened the chapter.

Kl: You know that, you're going to give us a test right? And you've been teaching us to learn in our groups, and the fact that every group has learned things differently,
Ro: Yeah. Different members.
Kl: And they've used different words, and they've used different ways of explaining it, and if you give us one test to everybody, that's not fair. Because everybody's learned it differently, used the words differently, and it's not exactly the same as, like I've learned it different from Monuel and he's learned it different from me.
MrM: And you're in different groups! But Monuel and Maria are in the same group, and they've learned differently, even though they've been in the same group. I agree with you. I should give you a test that catches you guys learning your way. Not catching you guys NOT learning something, because that's the old way that tests are supposed to be done, where I'm supposed to design a test that catches Ruby where she didn't know something or learn something. No. I should design a test that catches Ruby doing well. And that's hard. Because Ruby learned differently. Even though Rose who sits quite close to you is learning quite well in a completely different way. Now that's the kind of support that has to come

for new and innovative learning styles like you guys are being asked to do. A new and innovative testing style.

In the next (and last) portion of transcript, Maria accepts the invitation to think innovatively about testing for the inquiry they have been doing. She expresses her sense of vulnerability in regard to marks, and recommends that I address her need for assurance that she has learned what is needed. Then, she searches for a test that might match their prior learning, a short personal version of my profession's pursuit of 'authentic assessment' (Mathematical Sciences Education Board & National Research Council, 1993; National Council of Teachers of Mathematics, 1995). She closes with the group issue which she had broached first in the relative comfort of her small group, a week (and half a chapter) ago. Again it is very apparent that talk about assessment cuts closely to issues of great importance to students, in this case to the students' sense of vulnerability depending on a teacher's choices regarding test design.

Mr: I just want to tell you something about the test. Personally I think it's unfair, because everyone's been saying we've been learning differently, and we haven't really mastered, the ideas that we've learned. So for you to give us a test, grade nine's supposed to be a very important year for us. All of us are really quite afraid, we're afraid that we might bomb it, and some of us who want to do good, are going to go down. And I think it would be easier if you would go over again what we have learned, so we can remember it again, and learn some good things that we didn't get last time around, and maybe you can ask some questions in between our groups, like you could go, Okay you guys, if I give you a question similar to the ones that you guys have learned, now that we've done better, and we've reviewed it, maybe we can go about as a group, and do the question, and then come out with the answer. That would be more fair, rather than okay, you give us a test, and we all sit down and we go, okay, how do you do this question. Because all of us contribute to the group, so it would be really hard, individual tests, I think. And to me it sounds like a way to really bring down our marks, because we really don't understand this.

Ro: Understand it.

MrM: Two things. First, your marks shouldn't go down. I haven't had my head in the clouds all the time, and I've noticed how well you guys have worked. I've actually noticed how well you guys work and learn. I know darn well

I'm dealing with some very dedicated, capable people. I'm not going to say the word, smart, because that's not really all that important that you guys are so smart. You are, but it's no big deal. If I give a test, and the average mark is less than what you're getting in the regular math class, then I've got a problem with my test design. Period. I don't have a problem with you. If I have a class average of less than what you should get, then I've got a bad test, and I'll fix it. Period. Now that's half of it. Second, I agree, if you guys learn in groups then you should be tested in groups. Period. .

MsL: We also have, like 15 seconds.

MrM: I might steal the last fifteen, but I'll be around, if anybody wants. Why don't I give you out, something that could have been a test. And then I'll give you out something that could have been a test, two weeks ago. I've only got one copy for each group there, and that's probably a good thing. And I have one copy of a triple here, for each group. Take a look at these, and next time, if there is a next time, we'll talk about it.

In one sense, negotiating assessment can be viewed as a way to instill readiness for innovative assessment strategies. That was indeed an outcome in this circumstance. By posing the dilemma of assessing non-traditional instruction as a group inquiry, students came to view the dilemma as shared, rather than one which the teacher faces alone before making the students face the consequences of his decisions. However, underlying this process are some larger outcomes than just the alignment of students with the teacher in terms of assessment. When students talk meaningfully about assessment, they are talking about, and learning about, what counts as the outcome of learning, and what it means to them to be students/learners. They are engaging in epistemological and ontological inquiry. At the very least, they are hearing their classmates and their teacher framing epistemological and ontological questions and issues in terms which provide direct relevance to their ongoing classroom activity. "If we are serious about encouraging students to be mathematical meaning-makers, we should view the teacher and students as constituting an intellectual community. The classroom setting should be designed as much as possible to allow students to do their own negotiating and institutionalizing--in short, their own truth-making" (Cobb, 1989, p. 38).

Doesn't that sound wonderful? At the time, I wasn't feeling wonderful at all -- I wasn't at all sure if the students were actually with me at this point. Were the students even more uncomfortable than they were saying? I had encouraged them to talk about their concerns and what they would like changed, and they had done so: there was no talk about what they liked or appreciated or wanted to keep the same. The students were taking risks by expressing their deep concerns; I too was taking risks and as the conversation ended, I was not feeling secure at all. When students are dealing with pedagogy unlike their prior experiences and are likely to feel unsure about their success as they are learning, it seems appropriate to me that at times the teacher would feel similar ill ease. More important, a teacher must accept this kind of discomfort as an inherent cost of providing authenticity for teacher-student conversations by making the teacher's ongoing pedagogic choices subject to the direct influence of students. "A dialogic pedagogy, then, can only be transformative if dialogue means more than bringing students into the teacher script -- that is, 'giving students voice'" (Gutierrez, Rymes, & Larson, 1995, p. 18).

When I returned after the Christmas break, I quickly learned that the students were eager to proceed. Apparently, Maria's last turn in the December 13 conversation described well the extent of the anxiety of the students most concerned about the upcoming test. Their comfort level regarding mark vulnerability was moderated by my commitment not to let the test design lower the class average. Their concern for the match between learning style and test style was moderated by my commitment to design (somehow) a group-based test. Their concern for the way a test would treat the wide range of student orientations and the wide range of activity was moderated not by any commitment I made but my having listened to and validated this concern. More significantly, the students had asked for, and received, confirmation that my goals for them aligned with their goals for themselves. Now it was time for a test that could continue the growth which the December 13 conversation had made evident.

Negotiating the structure of the test

> Students need to see themselves as mathematicians in order to fully achieve the goals of thinking mathematically, valuing mathematics, and being confident and ready to use their mathematical expertise whenever appropriate. Feeling like an "insider" in the mathematics community is also crucial if students are to stop being intimidated and marginalized by school mathematics, "victims" of the decrees of "experts" in the field and of the myths that surround it (Borasi, 1992, p. 171).

As stated at the end of the transcript above, I had provided the students with some sheets (Appendix H). These sheets provided the framework for the testing decisions that followed. They provided an opportunity for me to address students' expressed desire to be shown what aspects of their learning processes were valued, and to help them recognize that they had indeed successfully learned from those processes. The sheets consisted of three pages which briefly revisited the four patterns activities the students had done to that time. With each pattern activity, the sheets asked them to describe and explain particular elements of the context. Rather than taking a comprehensive approach with each activity, the sheets sampled one element which the groups had likely encountered in their inquiry, and pointed to one element which extended the inquiry further in some way. In other words, the sheets directed students to retrace pathways they had already followed and to explore in a different direction than they had previously.

With each group I suggested possible pattern inquiries which could serve as test questions (Appendix I). My intention was to show possibilities for test questions that would ask students to do what they had learned to do well, which was to explore a pattern in a natural context. As students considered a sample question, they recognized that the situation itself was not complicated. Although the contexts of the new patterns were in no way like the contexts of the pattern situations they had already explored, the students recognized that they could begin with the same general attack strategies which they had used previously. Furthermore, they recognized that with the test design, they would not be accountable for discovering or deriving specific right-or-wrong answers, but for engaging well with the inquiry itself. This met with acceptance as a basic premise for the test, but as the discourse will show, students began to refer to this process as a "project" instead of a "test." Somehow, in stretching what would count as a test in my sense of the word, I had passed beyond what would count as a test in the students' sense of the word, without noticing the boundary.

I do not wish to emphasize the actual choices for review sheets and test questions, despite feeling some considerable pride in them as curriculum elements. The review sheets had questions that helped students to view again the algebra and the patterns from their previous inquiries, but the questions also helped the students to view again the processes by which their group had successfully inquired into those patterns. They were truly review questions. However, the sheets also had questions that invited the students to build beyond each pattern they had already recognized and to inquire further. These questions helped the students recognize that they had indeed built practical and applicable understandings and inquiry processes. Similarly, the test questions provided the students with a chance to recognize that they did have a general understanding of how to conduct these inquiries, and that they were developing a sense of the nature of the algebraic forms that they generated within each inquiry. Seeing themselves as competent explorers of new situations was a satisfying development.

To continue to challenge the students to consider what mattered to them in regard to assessment, I decided to give each group some choice in what approach to their test they would like. With four periods to use for reviewing and testing, each group could choice to spend 1, 2, or 3 period(s) reviewing, and, depending on their choice, they would receive a question that would be appropriate for exploring for 3, 2, or 1 period(s). Two groups found the review sheets' extension questions complex enough that they asked to make those their test questions, and I complied. I also decided that each group should decide what should be prioritized when I marked their inquiry endproducts. "Whether by student choice or by some form of negotiation, however, the outcome to be aimed for is that the students have a feeling of commitment to, and ownership of, the inquiry, so that they take responsibility for planning it and carrying through to a successful conclusion" (Wells, 1995, p. 23). Maria and her group were quite comfortable with this element, in part perhaps because of their proactive participation in the whole-class conversations about the program and how it could be assessed. As a result, their interactions with this process would serve only as evidence of ontological changes having occurred. To provide more clear examples of ontological changes as they are occurring, the focus will shift now to details of Kelsey, Ruby, and Colleen, struggling to make their choices about their assessment.

Students facing choices about their own assessment

> Real talk reaches deep into the experiences of each participant; it also draws on the analytical abilities of each. Conversation, as constructivists describe it, includes discourse and exploration, talking and listening, questions, argument, speculation, and sharing. ... In "real talk" domination is absent, reciprocity and cooperation are prominent (Belenky, Clinchy, Goldberger, & Tarule, 1986, p. 144).

This conversation provides an explicit view of the necessary interplay of ontological questions as the students dealt with the more purely epistemological question of what should serve as evidence of good learning. As they tried to determine what test might sponsor answers to the question, "How well have I done, as a student of mathematics learning by small-group inquiry?" Kelsey, Ruby, and Colleen were also concerned with the new consideration, "Who are we, to be asking such a question which teachers usually ask and answer?" As they perceived the school structure around them in a changing light, they perceived themselves as students somewhat differently. Equally important, Kelsey, Ruby, and Colleen will broach some issues in this conversation that became themes of inquiry for them. Two months later, these three students had done more than broach these issues, as the next chapter will report.

The transcript begins just as the group first begins to consider what shape their test will take. The group's initial reaction is to pursue a choice that will minimize their vulnerability. They do not express that they perceive any educational value to them in the test or the feedback it could sponsor; if they do perceive any such value, its significance to them is outweighed by their sense of vulnerability at being marked. However, Colleen does suggest that a written letter from me does provide useful information, and Mrs. Larkin encourages them to think about the possibility of truly finding out what they can capably do. Colleen would like to follow that up, but when she checks whether a mark from the choice they make might affect her report card, Mrs. Larkin's response does not simplify her dilemma: getting a mark from the teacher on the test might negatively affect her report card mark. As Mrs. Larkin keeps them pondering what matters to them, their image of tests as grounded in questions with right-and-wrong keeps emerging.

Cl: [Colleen] Okay so we're going to say what? That we want to do a project, or we don't?

Ke: [Kelsey] Okay, do you guys just want to go over everything and we don't want a mark, and we don't want a test? [laughing] or a project?

Cl: That could be our project, our project could be just trying to finish all up these xx.

Ke: Yeah and then we don't want a mark because we feel we don't need a mark. Right?

Rb: [Ruby] Exactly. If we want a letter, we can have a letter, instead of a mark.

Cl: Yes, a letter would be better.

Ke: Yeah, because, xx

Cl: Because a letter is more specific. It tells you in detail what you need to do.

Rb: Exactly.

Ke: Our project if we wanted a project, could be to finish up everything and understand everything.

Rb: Like go over everything we have done and figure out what we have been doing wrong and what we have been doing right and then feel confident in what we've done and understand it.

Ke: And then all he does is give us a letter on how we've been doing. And no mark.

MsL: [Mrs. Larkin] I can see that. I can also see that, maybe you might kind of add to that, you know, that doesn't sound like three periods. So maybe what you want to do is do that, and then say, Mr. Mason, give us a question that xx. Not as a test, but just as you've been doing.

Ke: And that is on a sheet. Like it will explain it, and it's not for any marks, it's just for,

MsL: Well, it could be for a letter grade, it could be how well, you might want to be marked on how well you attacked the problem, not on whether or not you got the right answer. Okay. Those are ideas, it doesn't have to be,

Cl: Right and wrong answers.

MsL: Yeah, it's not right and wrong answers, then, and it's not always just a final thing. It could be, how well are we progressing, it could be, how well are we progressing, how well are we learning, rather than, how much do we know, okay?

Ke: And then,

MsL: Does that help you?

Ke: Yeah. And so we could get, like, a letter not a grade, and,

Cl: Like the last one we did? Like the last letter we got? Or do we have to have a mark on our report card?

MsL: No, that hasn't been decided yet.

Cl: Is he giving a mark on our report card?

MsL: That hasn't been decided yet.

Cl: Well what if some people want,

MsL: You guys keep saying, basically, like this is what I hear you always asking, questions. Mrs. Larkin, Mr. Mason, tell us what to do. And we keep saying, you decide.

Ke: That's annoying.

Cl: Okay but what if, what if, what if

MsL: It's hard. It's something you're not used to, right?

Cl: What if some of the groups want a mark on the report card and some of them don't?

MsL: I don't know what we'll do about that. I don't have a definite plan about that, but we'll see how it goes. Okay? And then we'll talk about that.

At times, it became apparent to the three students that it is complicated being able to make decisions that are normally made by a teacher unilaterally. Below, Mrs. Larkin uses an analogy with them that suggests that real-life decisions are frequently made without a prior guarantee that there is a choice that is both correct and risk-free. The students must decide how much they are willing to give up, to make the assessment safe. As the conversation continues, Mrs. Larkin encourages them to view the process as worthwhile, and also ensures the students are considering all their options. Like wolf pups eager to leave the den, but afraid to leave the security it affords, the students hover at the edge of commitment.

Ke: Could we do a project, like take a project, and then get our mark for it and if we don't like it, like say we did really bad or something, then just decide you want an effort mark or something?

MsL: I don't know what do you think about that?

Rb: Suppose he gave us a question I don't know the answer.

MsL: You don't know the answer. Do you think that's fair? . You do something and then you decide, well that wasn't a very good idea, so now I don't want a mark. Do you think that's fair? . Yes or no, and why? .

Ke: I think it is, number one, we had those graphs, we did those graphs, and he let us redo them, because we didn't get the greatest mark on them, so if we could redo it and see what we did wrong.

MsL: So what you're saying is this is about learning and I'm xx. I have no problem with that. But you guys are always on about this is about right and wrong right? [laughs]

Cl: Well it's hard to, quit. Yeah.

MsL: Aren't you glad you don't have to do this every single day.

Cl: I think they should just, I think they should start right at kindergarten if they want us to switch us all over. And then we'll be already switched by the time we get to grade nine and then that would be better than trying to switch right through the middle.

MsL: Nobody's asking you to switch, in the sense of, this is not the way schools is going to be, but this is going to happen to you in school and in your life, right, that you are going to have to make some decisions, and whatever decisions you make, you are going to have to live with what it turns out like. Nobody knows all the right answers when they get married. Nobody knows that they picked the right person and it's going to be perfect and they're going to live happily ever after. But you've got to live with your decision, right. Nobody knows when they have kids if they're going to be perfect children. But you gotta live with it and make the best of it. And that's what you've got to do here. You've got to make some decisions and then you've got to live with them. I know that's tough cause you're not used to it. But this is a start, right. Are you great at this, probably not!

Cl: No!

MsL: How are you going to get better at it?

Ke: Practice.

MsL: Exactly. That's all [laughs]

Ke: Hoooo!

MsL: Talk it out. . What you have to get away from is Mrs. Larkin, or Mr. Mason, xx. Talk it out, and if you can support your decision, you're going to xx, whatever that decision is. Okay?

Ke: Yup!

MsL: We're here to help you through this process, not to grade you on it. Okay, we want you to become decision makers, we're not here to tell you you're failures at it. Because that's not a good way to learn anything. Okay? Does that help? Do you feel a little better about it now? . You get to decide, for once in your life, you get to decide.

Cl: I know, that's hard to get used to.

MsL: [laughs]

Cl: It doesn't happen a lot.

Ke: [laughs] I know it doesn't. "You're going to do this, and I don't want any talk back from you."

Cl: Or, "you're going to get a test on this day, and it's going to be like this, and you'd better be prepared. that's all we get. Not like, " you can make up your own test," and all this.

Ke: And then you went, "Whoaaaah!"

MsL: So it's a bad idea, to give you this experience?

Ke: No, it's good, it's just, it's new.

Cl: Very new.

MsL: And very hard.

Cl: Yeah.

Cl: It's different.

Ke: It's not hard, it's just different from everything else we've done.

MsL: Is it worthwhile for you do this kind of test, to see if you can learn from this?

Cl: Yeah. .

Ke: So are we supposed to, if we decide, okay we want to do, let's say, finish off things and get a new question, and then work that out. and when we work it out, if we don't exactly want a percent on it, not exactly, he shouldn't grade it on percent, xx

MsL: Why don't you say, um, you could even say to him, I want you to listen to our tape and tell us if we're going through a good learning process, or whether we're working well together. I want you to judge us, our mark, on how we work as a group, how we bring out the best in each other. That's the kind of thing. Whether or not we're asking the right questions, rather than getting the right answer.

Cl: Would a project be if he gave us like a totally new question?

MsL: Yeah, a project could be you could make up your own question. Or your project could be to make up a different question similar to Handshakes or Passways or something like that.

Cl: Yeah.

MsL: You could do that too. Your project could be that you make some kind of display about those questions, that we put up in the room that kind of summarizes what we've done. . Okay, you're on your own.

Ke: Okay.

Ultimately, the three students decided they wanted a project-style question, after they had sufficient time to go over the blue review sheets. In the portion of transcript which follows, they have called me over to tell me their decision and I let them know that I feel good about their choice. In fact, I felt good about the dilemmas which they faced when making their decision. To keep it going, I give them a new decision to consider. They suddenly find themselves thinking hard about what marks are for and what their value is. For safety considerations, they wish their mark could be based only on their effort, but they recognize that a mark should really be about more than that, and their sense of what good learning is guides them toward marking more than effort alone. Yet, whenever they feel confused, they retreat to the idea of being marked only on effort, like those wolf pups dashing back to the mouth of the den just to assure themselves that their sanctuary is still available. Students who feel too exposed, too vulnerable, will not choose to take the significant risks inherent in ontological questions related to their status as students. On the other hand, these students are certainly not in a risk-free context, for they recognize that their decision will be the decision they must live with. That lends a genuine edge to their choice of assessment criteria, a choice that to some extent means redefining their sense of being good students. Because their choice will be put into action, they cannot hold on to a comfortable yet unrealistic impression that assessment should be simultaneously risk-free and meaningful. "Learning is a complex, effortful, and often painful process. It can be exasperating too and also full of the wonder of new ideas and new skills. It can be painful to open one's mind, to change one's views, to try the unfamiliar. Doing such things is often threatening, even as they may be exciting" (Sizer, 1984, p. 150).

Ke: Can we give you our type of plan we're trying to figure out?
MrM: Did you talk about some different options first?
Ke: Yeah, we talked about two, two or three. Okay, we want to take a period and go over everything and tie up everything, and figure out,
Cl: Like from Odd Jobs and Passways and whatever.
Rb: Until we understand everything that we have been doing.
Ke: Pretty much. And then we want you to give us a problem that's similar to Handshakes, and Passways, and Odd Jobs.
MrM: Okay.
Ke: Similar. And then we figure it out. But we don't want it marked on the answer we want it marked on how we got the answer.
Rb: And the effort we put in and the learning.
Cl: Yeah.
Ke: And if we approached it in different ways to getting the answer.
MrM: Well that's what I think is important in this kind of learning right. So you and I agree on what's important. Like you just told me how it should be marked. Um, I give you a situation, and as a group you talk about it in a tape recorder and you write down some stuff and that's what gets marked.
Ke: Yup.
MrM: Is that what you had in mind?
Rb: Well we were thinking more along the line of what we did with Handshakes and Passways,
Cl: Yeah, something related to that, more like that than like the Painted Cube.

Ke: And more like what we did on the tape recorder. What, what,

MrM: Do you want me to just mark the paperwork, or do you want me to listen to the tape and mark that?

Ke: Well mark the,

MrM: Or do you want to mark the tape recorder? Like do you want to do the marking, that's okay?

Ke: We do the marking? .

MrM: Geez you guys, you're not willing to be creative, when you're thinking here, you keep wanting it to be like the old style of test!

Cl: But if we do the marking we can give ourselves any mark that we want.

Ke: Ahh, looks good, 90.

Rb: Yeah!

MrM: I didn't give you this promise when I walked in the door, I waited three months until I knew you guys, and then I made this promise. Do you really think that if I gave you a chance to just write down a number in my markbook, or in Mrs. Larkin's markbook, you would write down a hundred?

Cl: I wouldn't write down a hundred. Maybe, 85.

MrM: Do you think you deserve 85 in this process?

Cl: I would, it's hard,

Rb: It depends on what you mark about, because we put in a lot of effort, and we've been doing our best, but if we don't understand something,

Cl: Yeah.

Rb: Because, we don't, we put in a lot of effort, we're trying hard,

MrM: So I hear effort, and trying hard,

Rb: So if we're really trying hard, and we, if we think we deserve a different mark than you do,

MrM: And understanding,

Rb: xx.

MrM: Now how do we blend those three? How do we be fair? Effort, trying hard, and understanding.

Cl: What if we gave ourselves a mark, and you gave us a mark, and then we see how close they are, and if they're like, really close you could like average them and then if they're not really close you can just talk about why you gave us this mark and why we gave ourselves the mark that we did.

MrM: So we keep them both, we don't just average them together or something.

Rb: Well if there is like a really close difference.

Cl: Yeah, if it's really close you could, because then you know you've given us the same.

Rb: If we gave ourselves 90 and you gave us 60, then we'd,

Cl: Then we'd have to talk about it.

MrM: Or if it was the other way around.

Cl: Yeah.

Rb: Yeah, that's a hard thing, though, because we don't want to give marks too low that we'll fail. We don't want to give ourselves a mark that is so high that everyone says, Why did you get a mark like that? That'd never happen.

This is hard stuff! Without the right-or-wrong-answer style of test, where a percentage of right answers makes the percent score, the students are left without a formal derivation for the percent score. Suddenly, a question they hadn't asked before, let alone answered, presents itself: what do percentages mean, anyway? When Martin dealt with this question in the homework exchange (Chapter 6), he was not dealing with it in such a personal context as these learners were. Below, Ruby hopes that effort will prove a satisfactory answer, but I provide a sensible counter-case, and as I leave they begin a search for operational meanings for percentage and for understanding. Can the two be equated? Also, what about effort, anyway? Another idea they have taken for granted becomes problematic. They talk about something they are each very proud of as students, and realize their actual understanding of what makes their effort good, what makes their efforts pay off for them, elude their words. As suggested previously, the kind of effort involved in inquiry learning is very different from that involved in response to traditional instruction, in terms of initiative, dealing with ambiguity, developing strategies independently rather than by mimicry. Yet this group is only now beginning to question that there may be more to effort than just how much of it a student puts forth. The students begin to think differently about an element integral to their concepts of themselves as students.

MrM: So, let's design a test or a marking process, so that if anybody says, How did you get such a mark? We could say, look at this piece of paper. We designed this. This piece of paper gave us this mark.

Ke: You could have an effort mark. Percentages are fine and whatnot, they don't exactly tell percentages don't tell how hard you tried.

Cl: Percentages tell, yeah, they don't

Ke: The answer.

Cl: Yeah.

Rb: If you got the answer right., then you get this mark. But an effort mark says, that you tried really really hard and you did your best, so this is your reward.

MrM: Yeah, but there were kids, there were kids in grade eight who tried really really hard, right?

Cl: And they only got 50.

MrM: If they'd have got 90, they'd have been in this room. Right? Would they have done well in this room? I think they'd have done okay in our program, my program, but there's other math in this class with Mrs. Larkin, right, where you go through the book at a high speed. If they had been given a 90 just because they'd tried hard, they'd have got screwed because of that 90, right. they'd have been in a program they couldn't handle.

Rb: Yeah, but see that's the difference. That would be the effort mark. That's not the mark they got from knowing the stuff. There's a difference between knowing it and understanding the material.

MrM: Yeah, and we have to consider both don't we if we're going to put a score down. Actually, like, I figure somebody who's trying to figure what 's best for Ruby, or Colleen, or for, Kelsey. I figure somebody who's trying to design a program for you guys should care enough to read a three-page essay, written by you, and a three-page essay written by Mrs. Larkin and me, and then decide what kind of program you should have. It's just that's not what happens. They don't have the time to do that. They just want to look at a number and then decide what's best for you guys, right?

Ke: Yeah.

MrM: So somehow we have to find a number is actually what we would say in a six-page essay. That's a challenge. . Okay, get back to me.

Rb: Okay. How do you guys want to mark it?

Ke: I want him to write a letter, anyway.

Cl: Yeah. . . Okay, what should we mark ourselves on.

Rb: Effort.

Ke: Effort. Effort, understanding the material we've done, we've got to understand it.

Cl: Understanding the material, and what?

Ke: And how hard we like tried.
Cl: Isn't that effort and how hard you try the same thing?
Rb: Yeah, in a way.
Ke: No but, because how hard you try, you could try really really hard,
Cl: Yes it's the same. Yeah, how can you try really really hard, and not put in a good effort?
Rb: Okay.
Ke: Okay.
Cl: Effort, understanding the material, and?
Ke: And like, approaching it in different ways to get it,
Rb: Yeah.
Ke: Not just the straight common familiar ways to xx,
Rb: Xx see how many ways you can get it.
Cl: And then he could give us a mark.
Ke: And then we can make up a mark,
Cl: And then if there's a big difference we can talk about why there's a big difference, then we'll talk about why there's a difference, and if there isn't we'll average the two marks.
Ke: But don't tell him our mark until he gives us,
Rb: Is this going to be a group mark or is this going to be individual?
Ke: Group mark. Well all get the same mark because we're all working on it.
Cl: Yeah.
Rb: Okay.
Ke: Done. xx. Too many decisions.
Cl: It's kind of nice though, to decide when you're tested, what's on your test. If we could do that in social or French or something?

Too many decisions? The ultimate in having to make too many decisions is having to analyze deeply interwoven discourse such as this for predominant themes! The students' sense of vulnerability has remained relevant throughout the conversation. The shared sense that marks are operationally determined by right and wrong answers has faded, however, to be replaced by a triumvirate of *effort*, *understanding*, and *approaching it in different ways*. Although none of the three terms emerged from the students' conceptualization with a clear statement of taken-as-shared meaning, the first two have had elements of their previously assumed meanings questioned, and the third has evolved from a concern of just one student to become a shared topic. In terms of what counts as the outcome of learning, the students have problematized their entry assumptions admirably. Similarly, their senses of themselves as good students have been opened for reconsideration, not to send their confidence into retreat but to allow each of them to point confidently and proactively toward a new alignment of their studenting and their learning.

The students have posed relevant epistemological and ontological possibilities for further consideration, and are better able to reorient their view of learning and their view of themselves in the role of student to take fuller advantage of more aggressive ways to learn such as the ongoing inquiry into algebra. For these students, the question of what kind of test they should write and how it should be marked was a worthwhile question in its own right. This was a question which Maria had asked of herself, a question which she had posed as an interactive inquiry for herself and me. For Kelsey, Ruby, and Colleen, however, the question arose and the inquiry progressed only through the leadership and support of her teachers. However, it was also the starting point for them to engage in epistemological and ontological inquiry on their own accord. This dynamic and interesting group is featured in the next chapter, where they will speak for themselves regarding their progress from this point through the two months of the research period which followed this test.

For reasons outlined in Chapter 2, this study does not prioritize the testing elements of the instruction except as the tests themselves served instructional roles. As a result, the study will not provide extensive details about the tests which emerged from the negotiations and student decisions described above. In the end, for the educational objective of showing the students their capability as constructors of understanding through mathematical inquiry, each of the test questions (Appendix I) worked well with the particular groups who dealt with it. As elements of curriculum, I would not hesitate to recommend the use of any one of them; as elements of assessment, I recommend that anyone considering their use consider (and negotiate) how and to what ends they might be used with the particular students they are teaching.

Tests as mirrors for Maria

The decision-making and test performance of Maria's group will not be analyzed as was just done with Colleen, Ruby, and Kelsey. In part this is because it progressed remarkably smoothly: the students engaged confidently and purposefully, and as a result there was less evidence of ongoing ontological framing of questions and discourse.

Maria's group chose the three-period test question, because they felt they had learned so much that the shorter inquiries might not let them show how much they knew. Over the three periods, the students built grid-paper models, solved the question for specific values of the independent variable, found a sequence of solutions as the independent variable changed from 0 to 10, identified the increments between solutions and determined that it was a third-order relation, explained how changing the independent variable affected the solution, and algebraically expressed the relation. They developed their own extension question (what value for the independent variable would create the maximum of the function) and explained a trial-and-error approach that could answer it, and when I invited them to extend their set of independent variables outside of the domain of reasonable values, they were able to determine the shape of the algebraic relation's full graph. They got stuck lots, and improved their ability to work together. At one point, when they realized that they could do the graphs but that it would be time-consuming, they assigned it to themselves as homework. Their explorations and their development of products that showed their understandings were done as interdependent members of a single team. In other words, they did a remarkable job of a complicated inquiry, and autonomously made choices to maximize their learning. It is more appropriate to say "maximize their learning" here rather than "maximize their mark," even though they were doing a test, because the students had no direct impression of what would be worth marks, other than the guideline to show me how well they learned. In fact, all four members of this group ended up doing the test to show themselves how capable they were, more than to show me.

Each group received feedback on what elements they had developed well in their inquiry, and which elements they had not developed. They also received feedback on how their group functioned interpersonally, and, as mentioned earlier, I gave a percentile-like estimation of the group's success on the next letter to parents at report card time. However, for most of the students, my comments were only confirmations of something they had shown to themselves. The review questions that had let them view again what they had already explored showed them that they had truly made sense of the patterns within each context. The review questions that extended their investigations of those patterns showed them that they had developed working understandings of those patterns--in other words, they understood those patterns in ways that let them apply their sense of one pattern to their exploration of another. Finally, their work with a new question during the test showed them they had developed suitable attack skills and could see the interplay of algebraic expressions and the patterns which for them had generated those expressions.

When Maria was shown the rough draft of the letter to accompany the January report card, her comment in the margin captures this new awareness: "Mr. Mason, If you had given the math problem you assigned to our group to me in October, I would not honestly know where to begin. But now not only do I know where to begin, but now I can actually solve the problem which makes me feel quite good. Thanks!" For Maria (and her group), solving the problem gave them an opportunity to notice what they had been doing, *inquiring*, and what its effect had been, *learning*. "Learning to simplify is to climb on your own shoulders to be able to look down at what you have just done -- and then to represent it to yourself. The constructing or doing that precedes the new representation can be well or poorly designed. The good teacher is one who can construct exercises (or, better, provide experiences) that cry for representation" (Bruner, 1979, p. 101). Although Bruner meant the construction of mathematical representation when he said this, I am suggesting by its use here that this quotation applies to the learning that comes from self-representation as a learner of mathematics equally as well.

Although the test provided Maria with her best evidence, it was only a culmination of her many experiences and the meanings she had derived from those experiences over the preceding lessons. As her comment describes, Maria had constructed a different and more complex view of her learning and herself: she had been learning epistemologically and ontologically. To the extent that this learning was sponsored by the choices and activity the teacher provided and the teacher's interactions with her, then epistemological and ontological teaching had been taking place.

No one crucial act within the teaching described here could have convinced Maria to engage in personal inquiry through the assessment and ultimately perceive herself and her learning processes more competently. The students were worried about the test for a variety of reasons, but one which was easily expressed was that the test might unfairly lower their marks in mathematics for the term. Given that their efforts in the program have been sincere and hard-working, my promise that the test would not be allowed to have a class average less than they were receiving in the regular portion of the course was a simple enough commitment. Was it the pivotal reason in the students' willingness to take risks within the testing process sufficient to explain Maria's learning? I do not think so. Maria herself expressed that her concern for the test was more related to her sense of self than her marks. Listening to students such as Maria, in small groups, classroom discourse, and in the homework discourse helped to reduce their sense of risk and increase their willingness to engage in personal inquiry. By sharing with them some control over vital elements, the students' sense of vulnerability to the teacher's choices diminished, and perhaps their sense of personal possibility increased in turn. "Knowledge in more complex forms was possible when students' genuine relationships with authority replaced detached or apprentice relationships" (Baxter Magolda, 1992, p. 223).

Near the end of the research time-frame, Maria had a second opportunity to perceive herself with the help of a test, but this one was written individually. This data will close our extended relationship with Maria, and will provide a longitudinal perspective that will allow fuller claims about Maria's ongoing ontological learning.

The algebra tiles inquiry (Chapter 1) ended with the program's second test. (See Chapter 6.) Its circumstance was much simpler in terms of the dynamics of the relationship between the inquiry instruction (and me) and the students. The students were feeling little dissonance with their roles within inquiry learning, or with my role as their teacher and evaluator. Neither I nor the students were eager for a complicated or extensive process, nor was there a non-mathematical pedagogical purpose crying out for attention. I intended to bring some closure to the algebra tiles activity, and provide the students with a chance to perceive themselves as competent learners on an independent basis. However, I also wanted to help students to perceive the very personal nature of their learning, and I wanted to continue my own inquiry with regard to assessing within constructivist inquiry-based learning.

I designed the test to fit into one period, with one traditional-looking part, and one part not so traditional. In part one, individual students were asked four open questions which extended their work with the algebra tiles toward perfect squares. In other words, they were to show what they had already learned by learning more. First, students showed what they could do with $x^2 + 10x + 16$, an open question to let student show their basic understanding of factoring and whatever they perceived as connected to it. In the next two questions, students completed the polynomials, $x^2 + \_\_\_x + 16$ and $x^2 + 10x + \_\_\_$, their first formal experience with polynomial families. The fourth and final question asked them to identify the answers to the completed polynomials which would sensibly be called perfect squares, explain, and find some more (Appendix J).

All students were able to factor the polynomial in question one, sketch its rectangle and label its dimensions. Many students used grid multiplication to check their work. All students were able to find at least one more answer to complete each polynomial in questions two and three and factor it, and most found many solutions. Solutions using negative numbers were relatively rare. Some students found all the solutions, and some of these students claimed that they had found them all and justified their claim using a pattern or explanation. Most students correctly identified the perfect squares in their answers and were able to generate more. Again, some students were able to demonstrate that they had listed "all" of them, although some of these had not thought of negative values.

I felt very good about the test as an educational event but I found it very difficult to collapse into a single value all the information which it generated about each student's learning. In other words, it fit very well into my personal problematic about pedagogic assessment. As my response to each student shows (Appendix K), I did attempt to give students a numeric total for how many elements they showed and how many connections between elements they showed or explained. However, as I said to the students, I did not feel that the scores ranked them particularly well. More problematic for the students was that the score wasn't "out of" any particular total, since there was no total number of elements or connections that could have been shown or explained, and so it resisted their desire to simplify its meaning by making a percent. I posed this element of the assessment problematic to the students in a homework exchange, and below is Maria's response. The nature of my ongoing pursuit of understanding of assessment and inquiry is apparent in my reply to her, but it is Maria's response that is most relevant here. Mixed in with understatement I attribute in part to modesty and in part to her uncertainty about how to interpret all the information in the response sheet I provided, Maria shows and expresses her belief in the validity of her own independent sense of her performance, something that had developed well since that first test two months ago.

> Maria: March 11--*Homework: How should you get a mark out of 100 for the test?*
>
> Again, like the sheets, I believe I answered all the answers and even provided more than just the answer itself, like alternatives, diagrams, and even graphs. I truly think my mark should at least be over passing because I did provide correct answers on the test.
>
> *Of course your mark is over passing! It's at least a B! Here's my problem. A test should: a) help you learn more math; b) help you learn how you're learning; c) show you what the teacher wants; d) let you know generally if you're on track; e) make a score that shows how you compare to others.*
>
> *Teachers are great at (e), not bad at (c) (as long as they want simple stuff*

from students). This test tried to do (a), (b), and (c) for sure. And, in my opinion, it didn't do well at (e).

Thanks for the help.

I learned a lot from part one of the test, but part two was even more satisfying for me. Part one took only half the period. Then each student was paired with a student from a different learning group, to discuss what they had done on the test. Part two created no formal paperflow, but at the end of the period, each student was invited to write to their partner about his/her learning. I was very pleased with the results: students capably engaged with the ideas of learning and tests in various complex ways.

Here is Maria's response about Martin, her partner for part two, written the day of the test. You will note that Maria gives voice to the role of tests in confirming students' understanding. Like many of the students, Maria recognized and willingly discussed the role of confidence in their learning. As she was able to relate to me after the research had ended, before this test Maria believed that her group-mates would be able to do on their own as well as they did in the group, and she knew that she herself could not. She did not realize that the group dynamic contributed significantly to the performance of every member, not just to her performance. This test helped her to form a more appropriate relative impression. Not only did she perform independently in the test, but so did each of her classmates. This meant that Martin provided a realistic mirror for her, simultaneously serving as a comparison base and as an external source of an image of her learning. This written reflection is about her as much as about her temporary partner. In other words, the assessment played a significant part in her ontological development, as she saw herself well within her roles as a learner and a student, and interpreted those roles for herself. Of course, she also deals with epistemological considerations, as she decides what is valuable in her partner's (and, by comparison, her own) test responses. My reply attempts to be confirmatory regarding Maria's sense of herself, and lets her know that I would welcome further interaction, since the research program was drawing to a close.

> Maria: March 3 --*Homework: Say something about your partner's learning. Respond to their comment.*
>
> Martin was my partner for part two of the test. We discussed the answers we both had and noticed that there was a lot of similarities between the two. Discussing the answers made us both feel more reassured of the answers we had given. I believe Martin's knowledge of the information that we had learned over

the past few weeks is overall very good. His answers were straight to the point with little scrap notes. I think this is because Martin knows his information so well that he can quickly get the answers using his head. All in all I think Martin and I have a good understanding of the information, and I really enjoyed the nature of this test.

I am glad you enjoyed the test, Maria. It was a very aggressive design -- so aggressive, I didn't know how to mark part A until later and I have no clue how to mark part B. Yet, as you also feel, I feel it was a good learning experience for many of us.

I am glad you got some reassurance. Your courage and confidence is lower than it deserves to be, I think. Example -- how come whoever works with you GROWS so much? Sorry -- I diverge -- back to your comments. Talk to me if you wish.

I am intrigued by what you say about a good understanding leading straight to the point. I would agree, but somehow I feel that a great understanding includes seeing the COMPLEXITY of the topic. Can you help me with this idea?

Maria's description of what counts in her praise for Martin's learning is significant. Although I had designed the test to emphasize the application of their recently learned factoring skills to new processes, she expressed that the test indicated well how much of the unit's knowledge had already been learned. The test served Maria well within her more traditional sense of what good tests do, despite or perhaps because of its authentic assessment orientation. It is especially significant after the significant disequilibrium surrounding the previous test to note that Maria developed and expressed her sense of what a good student is and what a good student does with this test as well. The opportunity for students to confirm their ontological orientations as students is at least as vital to students' ontological learning as is experiences of disequilibrium.

Maria also expressed her sense that understanding something well is shown by being able to give answers quickly without using manipulatives or pencil and paper. This element was an aspect of her original more traditional orientation toward being a good learner or student, apparent in many students' earlier writing about what makes a good math student (Chapter 6). This suggests that Maria's learning about what counts in learning mathematics has not been a replacement of her original ways of thinking. Rather, she has integrated into that original orientation what she has learned by participating in small-group inquiry. Although she may have felt otherwise during her December crisis, to learn well and to perceive herself as learning well did not require Maria to abandon her entry orientation to learning or discard her prior sense of herself as an effective student and learner, something that would mean the rejection of the meanings she had developed from her prior experiences. Rather, she integrated into her original orientation elements that provided her with meaningful ways to see herself as successful in the different learning context of interactive inquiry. Just as any student's constructed understanding about a mathematical topic will include a student's prior mathematical understandings and experiences (Chapter 3), Maria constructed a new ontological understanding about herself grounded in and inclusive of her prior ontological understanding and experiences.

At the end of Chapter 6 is Martin's remarkable homework about Maria, written at the same time as Maria's writing above. Martin's homework was discussed in terms of how it reveals epistemological changes, learning about what learning is and how a particular learner can do it well, all within the framework of learning mathematics. Here, I have claimed that shifts in epistemological understanding will generally have ontological involvement, in terms of the students' sense of themselves as learners and as students, especially in relation to the teacher's role. If this is an important consideration, it must be one which is relevant to more than just the writing of one student. The consideration could also be applied to interpreting responses to part B of the test written by Wai, Kelsey, Ivan, and Hilde (see Chapter 6) in terms of the ontological learning of the writers as they describe the understanding of their partners.

Weaving ontology into the tapestry

Chapter 6 claimed that learning mathematics in more complex ways required that students reorient themselves epistemologically, and opportunities for students to construct such orientations are a necessary part of a curriculum based on an interactive inquiry approach. This chapter makes a corollary claim. Students' senses of themselves are dependent on and inclusive of how they perceive themselves as students, and that perception must undergo changes when they change the ways they understand understanding and learning. As a consequence, epistemological learning will necessarily sponsor students' construction of a more complex ontology, re-framing how they perceive themselves as learners. Again, curriculum must provide the opportunities for students to deal educationally with the disequilibrium that attends students' interactions with unfamiliar learning processes. The constructivist principles outlined in Chapter 3 provide an orientation for thinking about the provision of the activities and interactive reflective processes that can support students' continuous reconstruction of their senses of themselves both *as students* in relation to the classroom context as it could be viewed in institutional terms and *as people* in relation to the classroom context as a personal and interpersonal situation. This more complex view of mathematics curriculum suggests that students must learn about learning and understanding and they must learn about themselves and their view of themselves while they learn mathematics. The results can be that mathematics instruction would have value to students beyond the mathematics which they learn, and they can learn more mathematics more completely.

The next chapter will bring closure to the chronology of the instructional program. It also represents closure for readers' relationships with a group of students that brought into clear view the turmoil of reorienting one's approach to learning (Chapter 5). Ruby, Colleen, and Kelsey will provide their overall impressions of the instructional program and what it meant for them. As they do so, they will bring forward their concerns about being participants in the institutional sorting of students through their relative success in high school mathematics, concerns which interact with the themes of all the chapters to this point. The claim which the next chapter adds to what has already been said is that all the concerns that have been discussed so far (pedagogic, mathematical, epistemological, and ontological) are deeply interwoven with the complications inherent to the classroom and school system context. This provides a way to view the claims of the document as a whole and to situate them within the context which co-determined all of the outcomes described.

Chapter 8. The Final Facet: School Mathematics

This chapter highlights the voices of three students in an extended conversation with me at the close of the research intervention. Colleen, Kelsey, and Ruby look back on the interactive inquiry mathematics program and describe its meanings to them. They look forward to high school mathematics and beyond and connect their concerns about the future to their recent experiences. The conversation provides a retrospective of the major themes encountered so far, providing (it is hoped) a satisfying closure, interweaving of themes relevant to the factors which affected the learning throughout the interactive inquiry approach to algebra. As well, it brings into focus questions about the role of mathematics in school and how the school system manages the mathematics which the students do and learn. The chapter begins with concepts on which these new questions depend: school mathematics, the gap between educational research and educational practice which brought the term into general use, and the sorting function of school mathematics.

School math and classroom contexts

In contrast with mathematics as a discipline's ways of knowing and its objects of understanding, "school mathematics" is the translation of mathematics into subject matter for schools. Generally, the term is used by mathematicians and by educational theorists and researchers as a label of distinction, separating the mathematics which they understand as a coherent and active discipline from the mathematics enacted in school curricula and instruction (Atkinson 1992; Bishop 1988; Pinxten, 1994). As the quotes below show, the distinction is not made to praise what is perceived to be happening in the classroom. In the eyes of mathematicians and mathematics educators alike, school mathematics takes its nature from its school context at the expense of its mathematical content.

> The very nature of school math discourse is antithetical to the nature of mathematics as a creative discipline (Richards, 1991, p. 29).
>
> School mathematics tends to be defined "from the outside" and dominated by the examination system. Because of the socially defined claims on school mathematics, it can even be regarded as a subject distinct in itself; that is, distinct even from mathematics as practised in a scientific community. ... Teachers and pupils thus seem to 'connive together' both in insisting on a repetitive and highly structured curriculum and in avoiding the exposure of any differences in understanding or

> interpretation of a problem (Hoyles, 1988, pp. 156-157).

> School mathematics is a fixed set of facts and procedures for computing numerical and symbolic expressions to find determinant "answers." ... Given that view of content, teachers' central task is to provide clear, step-by-step demonstrations of each procedure, restate steps in response to student questions, provide adequate opportunities for students to practice the procedures, and offer specific corrective support when necessary (Smith, 1996, p. 390).

> [Twelfth grade] students perceive mathematics as a subject in which answers are right or wrong. They view the teacher as the authority figure whose responsibility is to pass on mathematics knowledge to students. "Doing mathematics" consists of memorizing rules and plugging new numbers into old formulas. Success in mathematics was defined as "getting good grades," which reflected a student's ability to quote rules and manipulate symbols and numbers (Miller, 1991, p. 516).

> The goal in the [middle school math] class was clearly not to learn through extended efforts but to get answers and complete assignments (Hart & Walker, 1993, p. 31).

> Task orientation involves a self-referenced definition of success as the gaining of insight or skill or accomplishing something that is personally challenging. Ego orientation, on the other hand, means that to experience success, the student must establish his or her ability as superior to that of others (Nicholls, Cobb, Yackel, Wood, & Wheatley, 1990, p. 140).

Mathematics as it is conceptualized and enacted in school is the result of a hybrid of intentions and students' interactions with school mathematics reflects that duality. The intentions of particular concern in this chapter are the dual school functions of educating and sorting students, with particular attention to the influence of the latter on the former, all within the context of mathematics at school. To help distinguish the actions of students in terms of their intentions in relation to these two functions, I will refer to students as *learning* when their actions are educationally motivated, and *studenting* when their actions are motivated for credentials or marks (Bereiter, 1990; Chickering & Reisser, 1993). "Children with *learning* goals want to comprehend; they accept the challenge of learning and persist in the face of difficulties. Those with *performance* goals think about getting the right answer and choose easy tasks as a way to demonstrate achievement. They also give up when success is not readily attained" (Holmes, 1990, pp. 101-102). In making a claim that the two distinctions (between educating and sorting, and between learning and studenting) are useful, I by no means wish to claim that the two are mutually exclusive or inherently distinct. However, as the quotes about school mathematics from educational researchers suggest, there is a gap to be addressed. (As well, the quotes themselves represent a gap between educational practice and educational research, an element that will be the focus of attention later in this introduction.)

I am using *sorting* to refer to the role of school mathematics grades for differentiating the opportunity access of students (Brewer, Rees, & Argys, 1996; British Columbia Ministry of Education, 1995; McEwen, 1995; Paquette, 1995; Reynolds, 1995), something of particular relevance to algebra (Bracey, 1995; Lewis, 1989; Niss, 1993; Noddings, 1995; Romagnano, 1994; Steen, 1992). Some critical theorists hold this role to be what determines the nature of school math, "the use of tests and examinations as means for control and selection" (Christiansen & Walther, 1986, p.245). Although more could be said to develop more richly the critical-theory issues of mathematics education as represented in the academic literature, to do so would detract from the grounding of this chapter in the voices of the students in the study. One quote must suffice: "The system is standardized so that all students in a class must learn the same things at the same rate so that norm-based grading can sort the students into laborers and managers" (Carr, Jenlink, & Reigeluth, 1995, p. 499).

As the opening quotations imply, reform-oriented research in secondary mathematics has had a difficult time incorporating the realities of school mathematics into its conceptualizations and interventions. Classrooms and schools are contexts where multiple elements insist on being relevant to the learning of mathematics, and those multiple elements resist research efforts to control or isolate them. As a result, much mathematics reform research to date has avoided the school context. However, an evolution of sorts is occurring in terms of math education researchers' relationships to classrooms. Acknowledging that the complexity of the mathematics classroom made it hard to achieve academic math goals and academic research goals has been a subtext of mathematics education research and theory for a long time (Thorndike et al., 1923). Sometimes this was acknowledged to explain the difficulties academics had in having positive influence on pre-college mathematics instruction; sometimes the acknowledgment was sympathetic commentary to precede well-intended advice for teachers about what they could do differently or better. Soon, there was more awareness of the depth of the challenge, first as a call to engage in reshaping school math through large-scale restructuring and now as an acknowledgment of the contextuality of classroom practice. This cascade of quotes demonstrates this transition.

> The first and perhaps the greatest disaster was the New Mathematics. .. Even though the program was based on valid theory and valid principles, it did not succeed because teachers were not properly prepared to deal with its subject matter, the theory was too abstract and not based on experience, there was a sudden break with accepted practice, and so on. ... The fundamental weakness of this and the other programs is that they are meant to produce specific cognitive, social and moral and affective outcomes, without paying heed to the specific contexts in which they are given (Eisenberg, 1995, p. 371).

> Mathematics classrooms must be restructured so that students' work in mathematics more closely resembles the work of mathematicians in the field. Our goal is to develop students who are challenged by messy, ill-defined situations or complex problems. ... Teachers embarking on mathematical restructuring should anticipate not only the excitement and enthusiasm but also the periods of confusion and feelings of incompetence that will occur (Parker, 1991, p. 443/449).

> The vast majority of the pupils could not provide any sensible justification for the permissible operations and it was obvious that for them these were no more than arbitrary 'rules of the game'. ... A real change for the better will not come until the teachers find ways to boost students' willingness to struggle for meaning (Sfard, 1994, p. 118).

> Constructivism, at least as it has been applied to mathematics education, has focused almost exclusively on the processes by which individual students actively construct their own mathematical realities. Much progress has been made on this front in recent years. However, far less attention has been given to the interpersonal or social aspects of mathematics learning and teaching. ... We will be unable to talk about the specifics of instruction in a theoretically grounded way unless we place analyses of learning within the context of classroom social interactions (Cobb, Yackel, & Wood, 1991, p. 163).

Focus and contextuality are tough decisions for researchers. The possibility of clearer outcomes and the possibility of outcomes whose forms are recognized by a broader community of scholars are dramatically increased if the complications of school mathematics (in its derogatory sense, mathematics as contaminated by elements related to student control and sorting) are held away from the research design and the focus is on mathematical and/or cognitive elements. Yet, the apparent applicability of one's findings to current instructional practice (for teachers) depends on its scope being inclusive of classroom contexts. This is similar to Schon's metaphor of the high hard ground of research-based theory compared to the swampy lowland of practice (Schon, 1983, p. 42). If only to contribute to teachers' perceptions of the credibility of mathematics education research as a general endeavor, I believe that more research on mathematics reform must accept the challenges of the complexity and context-specificity of classrooms.

Contextuality seems to be a forced choice between generalizability and applicability. Research which isn't structured to be deeply influenced by context may be assumed to be context-free, and this may be presented as yielding results which are context-independent, that is, generalizable. On the other hand, research which is deeply sensitive to the influence of a context (a particular classroom) can demonstrate its applicability in that one context and can incorporate into its design the possibility of influencing the context. Reports of such research can represent its outcomes as if they might have relevance elsewhere, but it cannot pretend to have automatic generalizability or transferability to other contexts. It can only sponsor and encourage the readers to consider transfer to their contexts by offering detail which supports comparison to the readers' contexts. Identifying and elaborating on common themes may honor context specificity more than generalizing for conclusions, but practitioners who are used to being told what to do by research will have to adapt to the invitational rather than instructional role which thematic and narrative research reports adopt. Chapter 2 described Cobb and colleagues, Borasi, and Romagnano as three sources of such reports.

This chapter will suggest that research about students' learning of mathematics can be sensitive to the impact of the school context on the students and their learning. Mathematics as perceived by students is inextricably embedded within the school context, so that reform of either content or context must affect and be affected by the other. An intended theme of the chapter a sense that the rewards of situating oneself within the school context can be worth the inherent complications. Recognizing that school mathematics is a hybrid of a discipline with a context provides an enrichment not a diminution of the research possibilities. Intended interventions can have greater impact and research findings can describe processes and results more realistically and completely, having acknowledged that students' sense of mathematics is deeply influenced by the context in which they learn.

This chapter also aims to bring together the threads of previous chapters, as a closing chapter must. It will be shy on reviewing the mathematics-based threads which served as the weft for this tapestry and which were featured in the first three chapters. However, it will portray a rich relationship the multiple goal-facets of mathematics education reform: more students learning more mathematics more completely; students perceiving mathematics and their learning of mathematics differently; students perceiving themselves as learners differently; and, finally, the emphasis of this chapter, students perceiving themselves as students in the school context differently.

Closure conversations as research technique

As the research was being brought to a close, the question of closure loomed. My original conception of closure was to have a set of closure interviews in which I would probe primarily for the students' personal cognitive senses of what algebra was, and then look at the data for themes and exceptions as a way to collapse the data across the class group. However, I realized that the research had been about much more than the students' understanding of algebra: it had been about the students' conceptions of mathematics, of learning, and of studenting. More important, the research had developed a different tone than originally conceived: the students had been encouraged not only to find voice but to direct the flow of the reflective elements of the research to their own interests and needs. It would make eminently more sense to hold closure conversations, rather than interviews. In other words, the students should be invited to talk with me, and should be encouraged to direct the flow of the conversation.

I wrote out a set of the questions I wanted to ask as a way to get prepared. This helped me to realize what my agenda might become, if the students and I slipped back into the standard stances of our roles. It made it possible for me to realize that my agenda in no case was of such significance, either in an educational sense in terms of my responsibility for the students' learning or in a research sense in terms of data flow, for me to need to pre-empt the students' agenda. I also realized how different were the questions that I wanted to put to each group of students and to each student: the deeply personal nature of what we had shared and what learning had occurred stood out even in that element. What we had each learned had been learned while we were together because we were learning with each other, but what we had each learned had been specific to each of us. This meant that rather than extracting themes by collapsing data across learners, my data interpretation should preserve the identities and uniqueness of each of the learners, a challenging but ultimately personally satisfying choice. In describing her research, Baxter Magolda describes a similar satisfaction:

> Despite many common themes, the interviews convinced me that these students' experiences cannot be described collectively. ... There are some prevailing winds, captured here as the epistemological reflection model, but the differences in various students' experiences prohibit placing them in static categories. Instead, the experiences generate possibilities for effective practice, one of which is to listen to students in particular practice settings for guidance (Baxter Magolda, 1992, p. xiv).

Almost all of the students wanted to have conversations and all asked that their conversations include their fellow group members. The three members of the group which included the student who hadn't provided permission for her data to be used weren't sure if they wanted a conversation, but they decided there was no point when I told them that what they said couldn't be used as research. Hilde, Silver, and Monuel decided that they were feeling very comfortable with everything, and that they didn't need to have a conversation. I let them know how much I valued their sense of what had gone on, but as Silver said, "With all we've been writing to you and all we've said to you all along, you want MORE?" How could I object? This left Maria out, and I suggested to Martin that he invite her to the extra conversation which he ended up wanting, and that worked out well. The other groups all scheduled one or two conversations with me, by arranging permission to miss a class or two with their regular teachers or scheduling the meeting for a noon-hour.

It was March. These grade nine students were choosing their high school courses, and, in many cases, choosing their high school. In this province, students could select academic mathematics (numbered as Math 10, Math 20, and Math 30 in grades ten, eleven, and twelve respectively), general mathematics in all three grades, or basic/remedial mathematics, although the decision was generally a direct consequence of grade nine mathematics marks. All of these students were going to take Math 10. The choice of high school was harder--they could select: separate (Roman Catholic) or public schools; a neighborhood school or one outside their neighborhood; schools with International Baccalaureate or Advanced Placement options; or schools with special programs and offerings. In every conversation, as the students expressed their concerns about these choices, they constantly returned to discussing what they had experienced or learned in the inquiry-learning program. The closure conversations did indeed provide rich data.

Reform and school mathematics

Colleen, Kelsey, and Ruby had a unique topic that they wanted to discuss. About a month before the closure conversation, I had shared an article from a local newspaper ("Math 30 gripes adding up," 1993) about some student and parent reaction to the provincial Ministry of Education's Math 30 final examination for semestered students that January. The ministry's final examinations determined half of all students' final marks in grade twelve subjects. Teachers, parents, and students were quoted in the article as being upset not just with the difficulty of the exam, but with its orientation. The examination authors had deliberately included some questions which were unlike the practice questions found in textbooks and previous examinations, questions on which most students across the province rely when preparing for the exam. For Kelsey, Colleen, and Ruby, this had excited their sense of vulnerability. In class, the article triggered conversation about whether my instruction for them or their previous instruction provided a better match with the orientation of the exam discussed in the article. Later, however, they realized that what really concerned them was whether the instruction they were to receive over the three years to come could be expected to match the exam they would write. I had offered them no assurances, especially since we were discussing an event to come in three years. The three students had decided that the orientation to learning which I had offered was good preparation for any exam where they had to think, but they were eager to talk about the political reality of being a mathematics student during a period of reform. In this portion of transcript from early in the conversation, Colleen introduces the issue, and the conversation quickly brings up the difficulty of changing things.

Cl: [Colleen] Do you think they're going to change the format by the time we get to grade twelve or before we get to grade twelve, change it to what will prepare us for this test?

MrM: Are you asking me are they going to change the test?

Cl: No are they going to change the program?

MrM: Well the program's already been changed officially.

Kl: [Kelsey] Are teachers going to change the way of teaching it?

Rb: [Ruby] Are they going to teach us to think instead of just learning how to memorize?

MrM: That sounds like one of my words, doesn't it?

Kl: Yes it does.

MrM: Yeah, I'm not a fan of memorizing, but I am a fan of thinking. I think it's a very difficult change for teachers to make, because to make that change is not just a matter of them changing the way they teach. It's a matter of convincing all the students that they should do something different.

Rb: You had a hard time convincing us!

Kl: Imagine how hard it is convincing people, who,

Cl: Who had three more years of being taught the same way.

Kl: Or aren't willing to change, cause we were,

Cl: I mean they're in their last year of high school, they get out after that. And if they go to university,

MrM: To be honest, I think if teachers wait until Math 30 and then try to change the ground rules, it might be very difficult,

Rb: Shouldn't they start changing in grade ten, though?

MrM: Maybe grade seven?

All: [hubbub]

Rb: If we're going to have a test, and let's say we never had this program this year, and we're going into high school, what if they started changing then, so people would think, then, would the test be easier? Would you know, would you be able to pass it, higher, if you were taught to think from the beginning of high school rather than just memorize?

MrM: I'm not sure if you need to be taught to think,

Kl: But you approach it in different ways.

Cl: yeah like,

In this context it is impossible to separate whatever anxiety may be attributed to the students' particular reaction to the article about the Math 30 exam scores or to any general sense of anxiety about school mathematics as a sorting system. Yet, here were three students, all well positioned to succeed at whatever mathematics came their way; yet their sense of vulnerability was significant. In fact, students frequently had unsettled me in different kinds of conversations, describing their sense that mathematics marks would control their future. Regardless of what else mathematics may be about, for these students, it was about qualifications, first qualifications for entry into and success in particular high school mathematics programs, then about qualifications for entry into and success in university, then about qualifications for entry into and success in occupations.

Yet it would be too hasty a conclusion to suggest that this portion of conversation was solely about the students' vulnerability to a test they couldn't control or predict. There is the issue of transition, of change, and the importance and difficulty of convincing students to accept change (and teachers--see Peirce & Stein, 1995, for their description of the "washback effect" of examinations on instruction). In the last transcript, Colleen brought university into the mix but didn't elaborate. Ruby described the theme of teaching to think versus teaching to memorize, which Kelsey interpreted as approaching a question in multiple different ways. Vulnerability to an exam was not an issue that the students perceived in isolation. In their incorporation of other themes into the conversation can be seen both the embeddedness of their concerns about school mathematics and the personal nature of the three students' concerns.

In the next portion of the conversation, all these themes reappear. When Colleen expresses a strongly instrumental sense of why she should take courses, Ruby and Kelsey suggest that there's more to life than courses. When Colleen concurs, Ruby brings back the role of courses and marks. Suddenly, in the midst of this, Colleen makes a self-descriptive statement, and suddenly the conversation turns epistemological. Yet, it never leaves the school context.

Cl: I don't have to take all these courses, but I want to take all these courses.
Rb: Yeah.
Cl: I want to take all the science courses, I want to take the good math courses, I want to take all this, because I want to learn, and I want to do something with my life, not just be a hamburger flipper,
Kl: And I want to have a chance to do things that I enjoy doing.
Rb: Yeah.
Kl: Like if you want to do photography, I want to do volleyball and basketball. I want time for that.
Rb: There's so many things you can do, and you only have three options.
Cl: There's so many things that I have to fit in to my life, like I want to fit in all these extracurricular activities, I want to fit in time with my family, I mean, I've got to fit that in somewhere. I gotta fit in all my homework, I gotta fit in all my classes,
Rb: But to do all this, you have to have good enough marks to get into university.
Cl: Because I want to fit in my veterinary career somewhere in my life.

Rb: And you have to pass that Math 30 exam to get into university.

MrM: You have to pass six or seven exams to get into university.

Rb: I'm just saying that's the most important one.

Cl: That's an important one, sciences, social, L A.

Rb: Everything's important, but that seems to be the hardest exam.

Cl: Yeah.

MrM: It might be that by the time you guys are in grade twelve, math has been scrambled around this year, but it might be that the teachers will have scrambled around some of the other subjects too.

Kl: I hope so.

Cl: I hope so, because science,

MrM: Right now that means that when you're in grade twelve there might be kids sitting beside you that are interviewed by the Journal to find out what's going on in Alberta's Bio 30 classes.

Cl: In Science I hope that we do lots of experiments, because I don't know, I just have this mind, this inquiring mind that has to know all this stuff, cause if I see something, I'm like, Does that work? Why is it doing that? It's not supposed to do that. That's like one of the laws of something or other, and it's not supposed to do that.

Kl: Something that you've memorized long ago.

Cl: Yeah, you've memorized all these laws, and you're out in university, and you see all these exceptions.

Kl: Like in French, why did we memorize all these stupid rules if there's so many exceptions?

Cl: Yeah, so many exceptions. Why didn't we just learn why some work, why some don't?

Kl: Yeah, French. You can do this for this verb, but not for this verb. That's the exception.

Cl: That's the exception.

Kl: And you can use this word only if you use this word in front of it. But if you don't you can't use it.

Rb: Or, this means first, this word means one. Now to use this spelling you have to have this verb in front of this verb.

Cl: All these exceptions, and I don't know why we bother memorizing, we should be learning why it is this way, and why it's supposed to be this verb in front of this verb.

If only Colleen had chosen traditional mathematics as her example space! "You can do this for this polynomial, but not for this polynomial." "Or, this means *x to the first*, this variable means *one x*. Now to use this notation you have to have a constant value in front of this variable." "I don't know why we bother memorizing, we should be learning why it is this way, and why it's supposed to be this term in front of this term." It would be stretching to conclude that the students had found their French instruction to be meaningless in comparison to their algebra instruction, or that they were able to perceive their French instruction critically because of their experiences in the inquiry-oriented algebra strand. However, it is quite clear that the boundaries between talking about mathematics as a school subject and talking about other subjects is not well-defined, and changing students' perceptions of one subject is likely to involve changing their perceptions of others, deliberately or incidentally. The students themselves determine the limits of an experience's relevance: the gateway between enriching mathematics specifically and challenging what counts as knowledge in school generally swings open (and closed) in both directions. Attempting to reform either could reform both (or neither).

The multiplicative nature of systematic change

One of the delightful and yet unwieldy facts of closure conversations as data is also clear in both of the above transcripts. Unity of theme and singularity of topic just doesn't exist! To my ears, the last transcript opened with statements of the students' sense of relative autonomy and sense of identity in their choices of courses (ontology). Similarly, Colleen's comment about her inquiring mind is deeply ontological. Yet it quickly becomes discussion about what counts as knowledge and how that knowledge should develop (epistemology). All of this is embedded in a discussion of courses, tests, and marks (the structure of school). My challenge as a researcher and as a writer is to isolate each of these elements only long enough to explicate it, before representing it as inherently interrelated with other elements. In dealing with closure conversations, interrelating is more natural than isolating, and I would claim that the natural state of these elements is represented well rather than poorly in such deeply intertwined and united flow.

Why do I claim that this contextual complexity is delightful? First, although it could be anticipated that the view of a changing element might be blocked by the presence of other elements, the opposite is more the case. The intertwined aspect of the elements makes the view of individual elements as they undergo changes clearer. Metaphorically, the light which is a byproduct of the chemical change in one part of the aggregate illuminates other parts, and the pattern of ripples through the other elements which results from any singular change in one element clarifies the elements' intersections. This illuminates the multiple elements under consideration, and also brings to light any elements which become relevant without having been included in the original conceptualization, even elements with only momentary relevance such as the declensions of French verbs.

However, the contextual complexity of mathematical learning is not an asset just for the observation of learning. The interdependent nature of the multiple elements affecting learning within a school classroom makes effecting change more possible rather than less. Although it is true that changing any one element is more difficult when other elements exist in a structure that tends to self-regulate its stasis (Bateson 1979; Maturana and Varela, 1987), when changes in multiple elements are pursued in conjunction, each aspect of change is supported and affected by the changes in other aspects, causing a greater overall effect. There is a virtually multiplicative effect among elements, where the impact of changes to the mathematics and how it is learned (pedagogy) are compounded by changes to the students' views of what counts as understanding and achieving understanding (epistemology) and by changes in the students' views of themselves as learners and students (ontology). This chapter, and this last portion of transcript in particular, suggest that the multiplicative interaction of effects carry over into, and from, issues of what counts as success in school and the structures of school which define that success.

Sense of self and reform

This does not mean that teachers and researchers cannot or should not attend to individual elements, those aspects of an interconnected reality which we declare as existing independently to make sense of the interdependent whole. The next portions of conversation provides an opportunity to isolate for further interpretation the relationship between marks and the students' sense of themselves. This relationship is useful to view and because, for Kelsey, Colleen, and Ruby, it is a relationship in transition. Perhaps teachers can or should only view learning as frogs view flies: it is through motion that the object of our attention shows itself. In this brief introductory piece the three students are characterizing what it means to be smart: Colleen uses marks and Ruby brings forward motivation and effort.

Kl: I know the difference between the maths.

MrM: If you want to be with smart kids you take Math 10.

Kl: Yeah but they're not exactly smart, because xx

Cl: It's not all smart kids, it's not like all eighties, but maybe that's better, because you won't feel so much pressure.

Rb: It's not all kids that want to learn, like there's kids that are smart enough and can try mostly.

MrM: You've said twice now talking about wanting to learn or learning, and you've also talked twice now about the value of the marks, and I think maybe one of the things you've got to do is decide whichever matters to you most.

Cl: Learning or marks.

Kl: They're both important.

Rb: They're both important, because if you don't have the marks, you can't get into certain programs at university.

Cl: Yeah and then,

MrM: You are going to have the marks, Ruby.

That's not much to go on! However, it provides a flavor of the students' values regarding what makes good students. The next portion of conversation provides more than a flavor. Of course that means other elements arise, but enough of how the students value marks come to view that distinctions can be drawn among the three students. The students are in transition in this regard, making statements that they would like to consider true to consider discarding orientations to marks which no longer suit their image of themselves. In my interactions with this process I make a claim about the relative reliability of grade nine marks in relation to grade twelve achievement, and I offer some direct (and honest) praise for these three students. Colleen, the least successful marks-wise of these three very successful students, states clearly the importance of her marks to her sense of herself.

Cl: I'm just going to believe what I want to believe, because I want to go to university.

Kl: And I don't want to think because, like I don't get the top marks in [illegible]

Rb: Me neither.

Cl: As long as I get marks that I'm proud of, they don't have to be top

Rb: Yeah, they don't have to be in the nineties.

Cl: No.

Rb: As long as you're proud of what you've done and you've worked hard

Cl: I'm just proud that all of my marks are above eighty right now, because on the first report card they weren't.

MrM: Well to be honest, I think of you guys, and I'll look at each of you guys, because when we talk about marks we're talking about individual marks, even though usually when I mark you , I mark you as a group. I do believe you guys are getting top marks, and I'll tell you why. I know you guys got, top 20 out of 100. That was your mark from me.

Cl: Mm-hmm.

MrM: And one group did get a much higher mark.

Kl: Top five out of a hundred.

MrM: Top five out of a hundred. And to me there's not a lot of difference, and what I mean by not a lot of difference, is by the end of grade twelve, or by the end of university if you want, but let's not make it that far in the future, by the end of grade twelve, so many things in your lives will change, that if we take these seven people together,

Cl: I'd like to see where we are.

MrM: You guys are just plain tops, so at the end of grade twelve, five out of seven of you guys will be at the very top in your class, and the other two will be in the top, but not the very top.

Cl: I don't think I'd want to be in the very top.

MrM: What I mean by that, is all seven of you will have the options you want.

Rb: I just want to be up there.

Cl: I just want to be able to see the top, you know. I don't want to be at the top, because that seems like too much pressure to me. Because I know that the people at the top like Silver and Hilde they have a lot of pressure to keep those marks, and they get really down if they lose only like one per cent. Me, one per cent, okay, I'll just make it up next time, no big deal.

Kl: Exactly. That's what it was for me, it's getting harder. I know what you're saying because my marks are all nineties, and when I get below a ninety I [illegible]

Rb: You freak! Kelsey, you go on a rampage! (laughs)

Al: Me, it's just below eighty because,

Rb: For me, I have a lot to do in my life, anyways. School is like, my main concern, but I do have a lot of other things going.

Al: That are just underneath it.

Rb: I can't be studying all the time. I have to be doing other things, so for me, if I get anything over 80 or anything over 90, I'm proud of myself.

Cl: Over 90, I'm so proud of myself, that I just go around and glow.

MrM: What does it mean when you get something over 90?

Kl: xx

Cl: It means different things to different people. To me, it means, because I know that my average marks are eighties, and when I get a ninety, that means that, it means, just that,

Kl: You're doing better than what your average is.

Cl: Yeah, that I'm for some reason, I was above average. It could have been because I was learning, I was listening I was learning more, or it could have been that the material was easier, I don't like to think that, but, it could be a lot of things, but I'm just really proud of myself when I get above ninety.

Colleen says it very well: great marks can be because of factors that are not related to who she is, but she doesn't like to think that. Rather, she wants to believe that marks are because she was learning successfully, or, perhaps more accurately, because she was a successful learner. The distinction is a matter of whether marks describe her actions or her self. "For some reason, I'm above average." I want to be careful not to read in to students' word use more than is there, but I think it is very significant that Colleen doesn't say, "my test answers are above average" or "my work is above average" or "my learning is above average." She thinks it is herself that gets marked. Colleen perceives the marks she gets as describing her, not just her work or her achievement. Many students talk about their marks this way: "I got a ___ on the test." Do all students conflate their identities with their achievement? Do teachers tend to encourage this? These questions shouldn't be considered answered by the speculative interpretations of the students' statements above, but they should be considered asked. Excellent students are easier to control when they conflate their identities with their school achievement, for it is the teacher who ultimately controls marks. Yet when excellent students conflate their identities with school marks, they are harder to teach excellently for it is the student who ultimately must direct their efforts for excellence to be the outcome. This brings into the picture the question of student autonomy and choice and teacher authority.

Reform and authority

Conversation about teacher authority will naturally centre on marks. In the portion of conversation which follows, Colleen, Ruby, and Kelsey trace their vulnerability to being controlled through marks to their teachers first, and then to their parents, and, finally, to the system. This is not the talk of committed rebels, or even of conscientious objectors. It is the talk of winners who are starting to wonder about the structure of the game they are winning. Their commitment to succeeding in school unshaken and perhaps even solidified by their musings, the three students are searching for an alternative to defining their success as students through marks. As they talk they find that their relationships with teachers, parents, and parents who are teachers incorporate a wide range of structures and orientations toward marks and learning.

> Cl: *I think that the teachers,*
>
> Rb: *They make it seem that marks are everything.*
>
> MrM: *Well you guys make it seem that,*
>
> Kl: *Well that's because how we're, how we're*
>
> Cl: *That's because of how the teachers taught us, but it's not exactly their fault, we kind of believed everything they said, [laughter] but we were taught to believe that, you know.*
>
> Kl: *Cause like grade six, I been always taught, marks, marks, and also my parents are both teachers,*
>
> Rb: *So are mine.*
>
> Kl: *So when they look at my report card, and your marks, that's all that counts. xx*
>
> Rb: *My mom, my mom is different. My mom is a teacher, my dad is a teacher, my mom looks at a report card and says Well this is good as long as you did your best. She really could care less. My dad is more, my dad is a high school teacher, he'll look at a report card and he'll say, well this is good, but it would have been nicer if you had had it a little bit higher, because he's a teacher, and that's how teachers teach kids. Marks.*

"That's how teachers teach kids. Marks." It's a pretty simple claim. Marks allow the teacher to acknowledge the success a student has achieved, but still suggest that the student strive for more. Marks are a language of communication of what is valued. In her last comment, Ruby switches from the orientation with which she started the portion of transcript, and describes it as a benign arrangement. Perhaps that is what brings Colleen to propose that marks can be a manipulative weapon in a teacher's hands, next. (In the remaining portions of discourse, I have changed subject areas and/or genders for teachers where those details are not directly relevant, as there is no reason to risk offending readers who could identify individuals from the context clues.]

Cl: But I think that when they're teaching us, and they're a bare-bones teacher, they're going to try to give us, like, the bare bones, right, and they're going to give us these really really high marks, right, but we won't learn anything.

Rb: Exactly. We're learning this much [holding thumb and forefinger 10 cm apart], and we have a mark like a hundred. Like in Social.

Cl: And we're getting a mark like, this high. [spreads her arms wide apart]

Rb: Like Social. We learn this much. [holds one hand up, thumb and forefinger 2 cm apart] We have a hundred.

MrM: A higher mark wouldn't have mattered unless they had made you addicted to marks first.

Kl: Exactly,

Rb: Well they did.

Cl: They did, and I've been, sorry Kelsey,

Kl: Sorry, Colleen.

Rb: Well I'll say something then. [laughter] Remember Mr. Lang, he always said, marks, no marks, if it was my way I would teach you no tests no marks no nothing, as long as you learn. We didn't learn anything from him, but that's a good idea.

Kl: We were sitting there, going, You're nuts.

Rb: But now when you come into a class, and you learn everything, and you take a test, and the marks don't matter but you pass the test, and you say, Hey, I learned everything on this test, like that's fine. But if you take a test and you don't pass and same thing, then fine, I've got to learn all this stuff.

My stance as a proactive member of the conversation rather than a neutral listener is exemplified by my introduction of the addiction metaphor in support of the direction the students were exploring. As in the homework exchanges, my role must be interactive and authentic to sponsor similar participation from the students. As in the homework exchanges, the students do not automatically accept my suggested directions or topics.

Ruby's last remark is a strong assertion different from her previous statements about marks: she and a test succeed whatever mark she gets, even if it is a failing mark, because the test tells her what she still has to learn. It is unlikely that Ruby had ever failed a test: I read this as a statement of how she wanted to react to tests to come than a description of how she had reacted in the past. Ruby is making a personal commitment to a new stance of relative independence from the emotional impact of test scores, committing to making it a learning event rather than a self-concept event when she gets a test back. Below, Colleen's response to Ruby suggests a more autonomous orientation than I had heard her express in any previous conversation, written or personal. Yet, it doesn't get developed, as her two colleagues indulge in a little marks-talk. Ruby expresses another aspect of her developing autonomy with regard to tests: she does not suggest that she cares less than before about tests, but that she uses more than the mark to decide about the meaning of her success on a test. Near the end of the transcript, I encourage them to use as an example their marks for their math "project," the January group investigation where they had asked for but not set up a conversation to get a precise percent mark rather than a somewhat vague one. The students make clear statements of the relative value of marks and learning which differ starkly from those made two months previously (Chapter 7).

Cl: I think I'm more proud of this stuff when I know I know. When I know I know all the stuff I need,
Rb: Rather than a hundred.
Cl: Rather than all the stuff they taught us.
Rb: Like in Science, I got such a good mark on that last test, I was so proud of myself.
Kl: I got an 87 and normally if I got an 87 it was a really hard test, I was so proud of myself,
Rb: I got an 89.
Kl: Usually I'll flip if I got an 87, but I was so proud of myself,
Rb: The second part of it I walked out of that room thinking, yes, I really aced that test, all that hard work, all that studying paid off, I was just so proud of myself. And then when I got the mark back, it was a bonus to see, you know. To get something that high on that test, you know.
MrM: Is that why we never had that conversation about the mark on that project?
Rb: Probably.
Cl: I think, that we thought about it,
Rb: We were mad, because we figured hey, we did this,
Cl: We were mad because we're still addicted to marks. Not so much before,
Kl: But we still are.
Rb: But still.
Cl: But there's still that little string that's connecting us between marks, and

maybe, we kind of think, if we get good marks, then we'll make lots of money.

Rb: Yeah, if you get good marks then you'll succeed in life,

Kl: That's like, that's what, that's like,

Rb: If you get bad marks well then you'll just be a hamburger flipper.

Kl: Your parents too. You NEED to get into university, you NEED to get into university. It's drilled into you.

Cl: But right now,

Rb: But no choice. Well you can take this course in university because you have low marks. You can't take this course, because you don't like it. That's what happens.

Cl: I'd like to be able to,

MrM: If you had wanted a mark that would have told you you were an 87 or an 89, from me, you could have had it.

Kl: I don't want it.

Rb: No.

Cl: No.

MrM: But not with the project you set up. If you had wanted a mark that distinguishes between an 87 and an 89, then,

Cl: Then you would have given us a test with just questions.

Rb: Yeah.

Cl: But I mean, that wouldn't,

Rb: But we learned more from that thing.

Cl: We learned more from that project than I think we would have from a test.

Rb: You can learn a lot more from being in class, than you can from a test.

The three students have learned. They recognize the relationship between mark structures and testing styles, an element which was relevant in prior conversations. That the students recognize marks systems as something embedded in instruction and testing procedures is important for them, as they proceed to develop their relative autonomy. Do these students ever find their growing autonomy regarding marks to conflict with teachers' authority? Even when Ruby suggested she can define her sense of success independently of the actual test mark, she acknowledged how much she cares about the mark itself. Below, the three students discuss disagreements about marks, and despite my attempt to steer the conversation toward learning, the students instead develop the idea that the authority of a teacher-marker shouldn't be absolute as a matter of principle. When they return to the inquiry instruction, they continue to develop the theme of student autonomy within the limits of teacher authority.

Kl: Our social studies teacher, she's the guidance counsellor, and
Rb: She's very close-minded, like you can't even argue a point on a test. She doesn't listen, and last year, our social studies teacher would listen to us, he would argue it back, and we would come to a decision ourselves, together you know.
Cl: Yeah.
MrM: You're making it sound like arguing about a mark on a test is,
Rb: No, I'm just saying,
MrM: Is a learning opportunity.
Rb: I'm just saying something.
MrM: You're not saying you're arguing about a mark on a test so your mark will change.
Kl: Yeah.
Cl: You're arguing about,
MrM: You're arguing to learn.
Rb: Exactly. Like if you have, like what we wanted to do with our mark. We were going to give ourselves a mark, You were going to give us a mark, and then we were going to talk about it together and figure out what we were going to do. That's learning. That's learning what you thought of us, and what we thought of each other, you know? But in a lot of our classes, they don't do that, they just figure that their word is law because they're the teacher.
Cl: Yeah, a lot of teachers think that.
MrM: So teachers think that their word is law,
Rb: Some of them.
MrM: And you say that makes it hard to learn.
Cl: Because I mean, if we were these nice little good good people and we accepted that everything the teacher said was right,
Kl: But that's not how it works.
Cl: Right. Teachers can be wrong.
Kl: Teachers are people.
Cl: We didn't used to realize that but now we're starting to, when we find a really obvious mistake and now we're starting to look a little deeper you

know, trying to find out, just to make sure that we're not getting gypped, these marks.

MrM: So marks become the situation that makes you fight back?

Rb: Well no not necessarily. It's A situation.

Kl: Or the fact that you have an opinion and she has an opinion and both opinions can be right, but hers is right.

Cl: But she doesn't let ours be right.

Kl: Yeah, ours is wrong because she's the teacher and she's right.

Rb: She has the teacher's book that says that she's right.

Cl: Exactly.

Kl: But our opinion doesn't count for anything. That's what I,

MrM: And you're getting treated differently by Mrs. Larkin and me.

Cl: Yes.

Rb: In this class, our opinion matters, and even when we were deciding whether or not we should take a test or a project, it was our opinion, our ideas.

Kl: What we wanted to do. Or how we wanted to tackle the problem.

This is an essential element for me in terms of research design. It was not an issue at any time whether I should be teacher-researcher as well as curriculum designer, in part simply because both Mrs. Larkin and I were comfortable with my performing those roles. Whether I should also be the person accountable for their assessment (within parameters acceptable to Mrs. Larkin of course) was a more delicate one. As the negotiation of instruction progressed, it turned out to be integral to the meaningfulness of the students' interactions with the instruction and with me that assessment and instruction roles were not separated. As the students' voices expressed in the above transcript, negotiations about what counted as far as instructional choices directly affected (and were affected by) what counted as far as marks. The richness of students' conversations with me, whether about marks or not, was enhanced by my having folded assessment into the instructional interactions. Given the students' initial orientations to school mathematics, to leave out the evaluative role from my profile with them would have meant that my instruction for them was less than real math.

As the conversation continues, the three students provide us with a retrospective on one of the crucial moments in the instruction and in the students' sense of risk and possible learning opportunities (December 13, Chapter 7). They describe a time when expressions of discomfort and dissonance were taken seriously, and the students suddenly found themselves with a chance to affect the kind of instruction they were to receive. As the students describe, it was more influence than they had had before, or had anticipated having, and they changed their orientations significantly at that time. Are the changes they describe attributable to the fact that they suddenly had influence, or to other elements that were part of that process?

MrM: That was the week after, I decided that if you guys, you three people in particular didn't want me here, I wouldn't be here. . . Do you remember that?
Kl: Yup.
MrM: I thought, either these guys are complaining about some things that needs to change, or else they're complaining because they don't think anybody's going to listen to them.
Kl: I figured, after that talk we had, I was glad we had that, because it got things out in the open,
Cl: Yup.
Kl: And it changed a lot.
Cl: Yeah, it did.
Kl: Yeah, it really did.
Cl: We started to be more open-minded, I think. We started,
Rb: Understand,
Kl: To understand what you were trying to do.
Cl: Yeah, before, we didn't understand what you were trying to teach us,
Kl: It was like, what are you doing!?
Cl: This isn't how we're supposed to learn this stuff, but now we realize that,
Kl: It's for our benefit.
Cl: It will help us in the long run. And, I enjoy it actually. [laughs]
MrM: What was the problem back then?
Kl: I didn't understand what you were trying to do.
Cl: Plus, we were as bad as the teachers, we were so close-minded.
Rb: We were close-minded, because we weren't used to.
Cl: And we wouldn't accept, yeah, change.
MrM: ... I don't think I said anything during that discussion that made you change, did I?
Kl: It made me think.
Cl: Yeah, it just by what everyone else is saying, well, maybe that's not exactly

fair what we're saying about him. Or maybe that's not exactly what he wants us to think or do, or,

Kl: I get that, now, I understand what you're trying to point out.

Cl: Yeah, suddenly this light bings on and we're like, Oh, this is what he's trying to do!

Kl: Like, I get it!

Cl: And then we just, it took us a while, we realized what you were trying to do, but we didn't accept it at first.

Why would close-mindedness become open-mindedness, all of a sudden? A mix of factors are thrown in by the three students: an understanding of my intentions, realization of the long-term benefits, realization of enjoyment. There was more than cognitive realizations being made. The students recognized that they were being listened to. Not only were their concerns being heard, but decisions were going to be shaped according to what they said. As a result, it was suddenly incumbent on them to be more reflective about what was occurring. The luxury of reacting negatively to something that was uncomfortable was complicated by a novel factor--what they said as they grumbled or complained would not only be heard but would be the basis for action that would significantly affect how they were taught. This is perhaps an awkward interaction effect: students who have no opportunity to effect changes in their environment have little cause to be judicious in their responses. They know they must react to the enacted decisions of the authority structure, but without the chance to influence those decisions or the structures, interactions cannot achieve one of its natural purposes, and the students are free to respond in whatever way they feel without affecting the world around them. If students are to interact in more complex ways with their instructional opportunities, they must have ways to influence those opportunities and recognize their efficacy to do so. When they can influence the flow of instruction by their interactions with it, their judgment must include the inspection of what they want as well as what they feel, and they can learn that considered responses can make a difference to their opportunities to learn and to succeed. Students can develop a sense of relative autonomy and the strategies to enact that autonomy when the authoring of the instruction is directly responsive to their input.

Reform and sense of learning

Marks are not the only vocabulary with which Kelsey, Colleen and Ruby can discuss learning. They also discussed the learning that they had been doing in terms of its personal value to them. In the first transcript of the chapter, the students had suggested that learning to think should be started earlier than grade twelve, as part of their discussion of the Math 30 provincial exam, and in the transcript below I was able to push for clarification of the idea. As the students describe the way they learned in the inquiry-oriented program, the issue of uncertainty and certainty arises in regard to answers, and (unfortunately, in retrospect) I wax metaphoric rather than letting the students continue. Although "learning to think" was not defined by the students and can be frustratingly ambiguous when used in discussing pedagogic priorities (Schrag, 1992), it frames a way to actively interpret the conversation itself: are the students describing learning to think? Is a conversation like this a part of learning to think?

MrM: I think, the idea, now you're bringing up another idea, and that's we're talking about teaching to think, and now you're talking about teaching to think in more than one way, and those are big ideas. I think in general you will find, for example, if you're watching what I've done this year, if you add up what I do in a period, I'm kind of useless. Right? When I'm here visiting you guys, I'm kind of useless.

Cl: We do a lot of talking but, like, in the old learning, we don't, like,

Kl: We don't talk, we don't even work as a group, you figure it out yourself, you do it yourself,

Rb: So you figure it out, and you get the answer,

Kl: You get the answer, and you go on.

Rb: You say, oh I got the answer right, next question. Or oh I got the answer wrong, and you go back figure out what you got wrong.

Cl: Oh but most of the time people don't figure out what they do wrong. They just say Oh I got the answer wrong, write in the right answer, and go on to the next question.

Kl: In this set of learning, we do it, and then we don't get the answer usually right away.

Cl: We don't usually! Sometimes we don't get the answer at all!

Rb: And if we did then we don't know if we were doing it right cause we don't have the answer. Is that what Math 30 is going to be sort of like? And there is maybe more than one different answer, and you have to do it a certain way?

MrM: I think one of the things that has to happen before you guys become my age, is you're going to have to figure out how to decide on your own if something is right or wrong. Like here I am trying to decide how to answer your question, Ruby, and I'm never going to find out once I answer whether I gave you the right answer or not. I'm not going to look you in the eye, and you're going to say, Okay Mr. Mason, I'll give you a 7 out of 12 for that.

Rb: 3 out of 12. [laughter]

MrM: So as an adult you have to go through life with all these complicated questions, and I don't get to find out, I don't get to look up the answers. There isn't a textbook called life, and look in the back and see if I got today right or not!

Kl: I wish it was like that!

Rb: Yeah no kidding!

MrM: Do you really wish it was that simple?

Kl: Sometimes.

Rb: Sometimes.

MrM: I mean it's definitely that simple for a carrot, right. As a carrot, if he looks up the answer in his owner's manual, let's see, Sit there, wait as a seed until something makes you wet. And he says, oh good, that's exactly what I did today. One out of one. And then he waits for tomorrow, and tomorrow, at the end of the day, he flips in his owner's manual, looks to the back for March 2, 1993, and says, I grew towards the sun today, and I put my roots downward. How did I do? Ah, 2 out of 2 for today. So carrots have it easy. And I'm suggesting to you that you don't really wish it was easy.

Kl: Sometimes.

Cl: Sometimes we think we wish it was easy! [laughs]

Here is a question which math teachers must face: should Math 30, or Math 9, or any math, be simple or easy? As you heard in previous chapters, these three students believe that they should work hard in math class, but that expression of industriousness is not equivalent to saying that math class should be hard, as in difficult. Certainly if comfort and good marks are the desired outcomes of a class, the students will demand of teachers that they do what they can to make the "work" easy (J. Mason, 1989; Schifter & Fosnot, 1993; Schoen & Hallas, 1993; Weiss, 1995; Wilson, Fernandez, & Hadaway, 1993). Performing drill questions which are similar in kind to demonstrated examples is intellectually easy, and an instructional relationship which is predicated around telling and demonstrating, assigning practice, and testing by providing questions like the practice questions is an easy relationship for everyone involved (Brissenden, 1988, Sizer, 1984). Inquiry, even purposeful, focused, and structured inquiry, is not. Yet, learning to think and thinking to learn are less available to students when performing drill questions.

I wonder how much this conflict represents a partial answer to the pondering which opened Chapter 3--why don't more math teachers create inquiry environments with tools like algebra tiles? Perhaps it is because they accept that their role to help students is equivalent to the image of making things *easier* for students, making it *easier* for students to do their "work," *simplifying* the relationship between work and marks. If so, then establishing inquiry as a dominant learning strategy is a violation of that orientation. In a parallel sense, if teachers believe in helping students to be comfortable, then perhaps it would be difficult to provide activity which, by its exploratory nature, includes the uncertainty, even discomfort, which Ruby has described.

In another sense, however, Ruby has expressed an element of liberation from a particular authority which teachers (and texts) wield in traditional instruction. When answering a question such as the product of two binomials, a student must submit that answer to a teacher or to a published list of answers to determine its correctness. That is more than a dependence on the teacher and the text for a particular service. It is symptomatic of a question in isolation of any context in which the sense of the answer can be judged. In contrast, when inquiry takes place in a context in which the mathematics being investigated makes sense, the specific answers which students generate (and by extension, the methods which students use) can be seen to be sensible. For example, when two binomials being multiplied arithmetically are also the edges of a rectangle which students have created, students can check their arithmetically obtained product with the area of the rectangle. For these students, a calculation process (a standard algorithm or the grid arithmetic of Chapter 3) does not simply generate or check an answer: it confirms the sense-making which generated the answer and connects the contextual understanding to the symbolic structures of mathematics. In turn, this means that the students can proceed more autonomously, not only making sense through inquiry but monitoring their sense-making, and perhaps in reflecting on the processes discovering the complexity which Ruby has described. What is an epistemological revelation (answers can make sense by considering them within the natural context of the question) becomes an ontological transition (students can construct and confirm their understandings without complete dependence upon the school structure's authorities to authenticate their answers and processes).

Is inquiry a way to learn mathematics (Borasi, 1992; Bruner, 1960/1995) then, or is it the nature of mathematics (Rogers, 1994; Schifter, 1996)? Is the importance of learning mathematics situated in its relationship with learning to think? Colleen, Ruby, and Kelsey cannot ask these questions. They are firm believers in inquiry as a way to learn mathematics, but they have not made the leap to wonder whether that is what should truly count as mathematics. They have an operational definition of the nature of mathematics, and its embodiment is the test at the end of grade twelve. In this conversation they repeatedly have expressed that ideally the test style should match the instructional style, but they have not suggested that the test should match (or portray) deeper underlying truths about its subject. Waxing philosophical about underlying truths seems to be a privilege of those like me whose success is not defined by a prescribed endproduct.

Reform and learning to learn

What do the students say about learning? Generally the students' statements about what they learned and how they learned in the research strand is through contrasts with other contexts, other classrooms past and present. The constructivist premise that students understand by (in part) connecting new experiences to prior ones suggests that this should not be surprising. At one point, I asked directly what their learning processes were this year in mathematics. The conversation switches quickly from considering the cognition of learning to considering the overall classroom interactions and student activity, as if "learning" was supposed to point toward the overt activities of students in a classroom. At the end, Ruby switches the focus back to the cognitive sense of the word.

MrM: What did learning look like to you last year?
Kl: Learning was memorizing.
Rb: Math wasn't exactly the best example from last year.
Kl: Because this is what Mrs. Smith did. Here's your homework, kay, walk in, sit down, correct your homework, here's your homework and we'd be done twenty minutes before class ended.
Rb: Yeah, anyone who needs help come to the front of the room.
Kl: Yeah, that's it.
MrM: And did people come to the front of the room?
Kl: No, because nobody wanted to be like,
Rb: Not really. Because everyone pretty much knew what was going on.
Cl: No because we were sitting in groups and we would help each other,

remember? You me and Rhoda.

Rb: But we didn't learn anything, we learned it on our own, but we didn't really learn how to do it, we just learned by memorization and reading the book.

Kl: We learned one way to do it.

It is quite possible that these students are much like other students in accepting that teachers' use of the word "learning" usually points without consideration of cognitive progress to the general activities going on in a classroom. On that level, having learned in a different way during the inquiry program has provided a framework for noticing the ways that learning proceeded previously, and distinctions become possible. In a written response to a similar prompt early in the year (October 7), Ruby made similar comparisons. However, she uses "learn" in its more distinctive role at the end of the response: "This learning is different than from the past because in the past we just had to write down the answer. Now we have to explain what we are doing, how to get the answer and other possible ways to get the answer. We also come in and learn something every class rather than coming in, finding out our homework and doing it." Ruby made the same transition in the use of "learn" in the conversation above, and even provided a layer about learning how to learn.

In the next portion of transcript, the students return to describing recipient-style learning. They suggest that the role of that approach is likely to be discipline-specific. However, they also talk about the internal actions associated with learning. "Memorize" gets described in the frame of recipient-style learning by what it is not, not "daydreaming or looking around" and not "let it all pass right through." "Retaining" is described as an internal process to be equated with "absorbing." Despite their sense that this is how to do well in many subjects, they are unequivocal in stating that learning in the mathematics activities we shared involves much more.

Kl: Except in grade ten it's different. They approach, the teachers, I was talking to my dad he's a high school teacher, they approach everything totally different. They don't treat you like little kids.

Rb: No, that's not necessarily true. [Ruby's parents are teachers as well.]

Kl: They're like, okay, here's what you need to know, here's a book, it's due in three weeks, have fun.

Rb: That's not necessarily true either.

Kl: And they teach you what you need to know.

Rb: It depends on the teacher.

Cl: And it depends on the subject, because, science, I don't think you can do that. But maybe social you can, because if you're doing history, you read, you have to memorize dates, or that's what we have to do now, and then you get tested on it.

MrM: Suppose in science class, they do teach you every day, and then they give you a test.

Cl: I'd like that.

MrM: If the teaching happened every day, wouldn't everybody get the same mark on the test?

Rb: No. It depends on how they were learning, xx

Kl: No, because people xx, if they were daydreaming or looking around, or sitting there sleeping of course they're not going to do well.

Cl: Or they can just let it all pass right through them,

Rb: Go in one ear and out the other.

Cl: Yeah, or they can retain it.

MrM: What does retaining look like?

Kl: Retaining doesn't look like anything, it's what you do personally. I could sit here, and go like this, and listen to you talk, and nod your head, but I wouldn't understand a thing you say.

Cl: Like I could hear you, but I wouldn't understand you, because I wouldn't be listening I wouldn't be learning.

Rb: Or you could just sit there and not say anything the whole time, and listen to everything that's going on,

Cl: And just ABSORB it.

Rb: And just absorb it all, and then pass the test with really high marks.

Kl: Or I could sit there, and be in a daze, half there, listening, or not, or I could sit here and listen, and add my own comments, and listen to what you have to say, and that's how people learn differently.

With passive learning, then, there are two ways to get "really high marks." It is possible just to absorb by listening. Others will add their own comments, staying involved by doing so. However, both are dependent on Colleen's distinction between hearing and listening. With success defined as getting good marks, I am suspicious that non-cognitive senses of "learn" were implied during the conversation. Regardless, how relevant is passive learning to the inquiry program, and mathematics? It is toward that question that I direct the students' attention below. I like their response very much. On one level, the students provide a description of constructivism within inquiry activities. Notice also their continuing use of the phrase, "We had to ..." which suggests (unless it is just the continuation of their natural phrasing for describing what students do, evident in their earlier descriptions of other classrooms) that the activities which they did were set for them and that the deep level of involvement was also clearly expected. They close the passage by making reference to the tests, responding in ways that depend on what is valued as valuable learning outcomes. It's a complex little passage, rich with overlap between what counts in mathematics, what learning is about, and how studenting and learning can co-emerge as something more complex and more empowering.

> MrM: Could you have done the program I did with you, by absorbing and listening?
>
> Kl: No. Not a chance.
>
> MrM: What's the problem?
>
> Kl: What you gave us was things we had to work with our hands, we had to draw things, we had to make things,
>
> Cl: It's kinda like molding play-dough with our brains. That's what we had to do. We had to turn things over and over, and knead it, and try to squeeze out the information,
>
> Kl: And look at it in different ways, It's like okay we could get the answer this way,
>
> Cl: And then kind of put it all together like a puzzle.
>
> Kl: We could get the answer this way, but if you look at it this way, you could also do this!
>
> Cl: It looks kind of different you know and you want to see how you can get that,
>
> Kl: And then you turn it upside down, and it could be a whole new way to do that!
>
> MrM: Couldn't Ruby have just sat there watching you guys turn it and twist it and make it?
>
> Kl: You have to look at it,
>
> Cl: She could have, but, that wouldn't have accomplished anything.
>
> Rb: Well, yeah, that's one way. Yeah, it could have.
>
> Cl: It depends on what we're being tested on, because if we were being tested on HOW to twist it and turn it, or if we were just being tested on just the

answer, but twisting and turning?

Kl: Exactly.

Cl: She would have just seen us do it, she wouldn't have known HOW to do it. She'd just kind of look at it, Oh great they're just twisting and turning, that's great. [laughter]

Molding play-dough with our brains. This is a remarkable visual metaphor for constructing imagery from experiences. When Colleen suggests that after perceiving different views learners must, "Kind of put it all together like a puzzle," I feel she captures the synthesis of elements into a recognizable (re-cognizable?) whole. This provides a real purpose for Kelsey's often repeated theme of many ways to the same answer (Chapters 5 and 7): it's not just what the teacher has set as the new rules of the game, but a necessary prerequisite of synthesis. "You want to see how you can get that." For me this evokes the natural sense of purpose in inquiry, and the sentence stem is a welcome relief for me after all the "We had to..." which dominated this section's first few lines. Overall, I wish only that the conversation had paid some homage to the role of discourse--I think it odd that a group that was so talkative seems to have taken for granted the role of conversation in inquiry. Regardless, the conversation demonstrates well that they are also molding their molding of play-dough with their brains, to construct richer imagery about their own learning.

Reforming what students think about mathematics

As mentioned earlier, a closure conversation isn't an appropriate process for measuring the mathematical achievement outcomes of a program. Yet to the extent that beliefs and feelings about mathematics are relevant to mathematical achievement, a closure conversation can provide significant evidence. Ideally what should students believe and feel about mathematics upon entering high school? Colleen, Kelsey, and Ruby provide personal descriptions of their orientations to mathematics as a subject. Again, marks provide a vocabulary for such a conversation, and by making forced choices between good marks and other possible outcomes, the students express their values in ways which we can interpret. In the next portion of conversation, the students bring forward three goals which they hope are not exclusive, and I ask them to interpret those goals in relation to mathematics.

Cl: Like right now I'm taking a self-defense course, and it's really good, but that's ending this week, and I might take another course, but they're so expensive.

MrM: And what mark are you going to get?

Cl: Oh, we don't get marks.

MrM: You're taking a course where you don't get marks?

Cl: Yeah. Self-defense, just at the community league. Me and Silver, there's belts, too, that we can go for, so we might go, like that.

MrM: Is that going to get you into university, or,

Kl: No.

Cl: No.

Kl: That's something to do. My volleyball and basketball, it's not going to help me get into university.

Cl: Exactly.

Rb: xx but I enjoy doing that.

Cl: Yeah. It's something we enjoy doing. If we don't do anything in our lives that we like, enjoy, if we don't do anything that we enjoy, then we're going to have a pretty boring life.

Rb: We'd be bored.

Cl: We might make lots of money, but we wouldn't be happy.

MrM: Okay, so now we have to get this choice broadened now, between marks and learning and enjoyment.

Cl: Can't we have some of each?

Rb: Exactly.

Cl: I think we could. I think we could learn, and get good marks, and we could enjoy ourselves.

Rb: We could have all three.

Kl: That's very hard, that's very hard.

MrM: If I'm going to help you a lot, the most help I'm going to give you, we have to broaden the discussion sometimes, right, but if we bring it right back down to what I'm an expert at,

Kl: Math.

MrM: And what you guys are an expert at, then we can have a first quality conversation. We can broaden again, we have to, and then focus again. Can math be about marks and learning and enjoyment?

Cl: I enjoy math. I look forward to it a lot, I like it.

Kl: I like it because I learn something.

Cl: I learn something, and I'm able to get good marks in it, and I enjoy it just because it challenges me. Like xx

This kind of talk is likely transitional or exploratory in purpose. From discussions about mathematics success which is defined purely within the frame of the institution by marks and achievement, the students have begun to move toward defining mathematics success differently. However, they were not yet positioned to privilege their own sense of what counts as mathematics learning separate from the institution's structured measuring of what counts. As a result, their talk extended toward broader conceptualizations somewhat as trial balloons, before retreating to prior conceptualizations. New connections were expressed and ideas tried out, such as the idea that learning in itself makes an activity likable, or that challenge makes something likable. This is much broader than statements that students like something where they earn good marks relatively easily. In the last comment above, not only was Colleen trying out an idea about mathematics that may have been new to her, that it is enjoyable, but she was also considering why math was appealing to her. The use of "challenging" contrasts meaningfully with "frustrating," the word she used much earlier in a written response. "I worked really hard and completed all my assignments. I have a good attitude toward this, even though it's frustrating (October 20)." Is the difference between feeling frustrated with one's work and challenged with one's learning a difference in goals, or confidence, or both? If lack of confidence constricts the range of a student's zone of proximal development (Vygotsky, 1977), they will less often be prepared to engage with inquiry tasks. Could the range of a student's ZPD be a more significant factor than its location along a scale of prior knowledge? Surely appreciating and enjoying challenge is a positive asset for beginning high school mathematics, and would be a challenging goal for mathematics instruction to accept as a priority.

If a positive attitude toward the subject area and toward learning is a desirable outcome, it can only be added to, rather than replace, the cognitive outcomes. In the portion of conversation below, I use the language of students in school, final exams, to cut to the quick of the matter. What do these students think grade nine math is about? Although their answer stays very general, and ends up being more about qualifications than about content, they make a forced choice between, in their words, "memorizing" questions and "thinking" questions for their final exam.

MrM: What would the grade nine final exam in math look like to you guys?
Cl: If we had a choice?

MrM: What would your math final exam look like if I really let you guys plan it?

Cl: Well I think there would be some memorizing, because that's still the way the world works right now.

Rb: Like for certain things, like if you want to find the circumference of a circle, there would be a memorized formula.

Cl: But there would also be a section with some of those questions that you gave us. That might be part two, we might have to do it in two phases, and one might be just kind of,

Kl: Like some thinking questions, some questions you have to,

MrM: Even though that means you can't have an accurate mark.

Rb: Well that would be better.

Cl: Yeah.

MrM: Because if you start doing project type exams, there's no way I can give you an accurate mark. I can tell you guys you were in the top fifth of the world, but,

Rb: Who needs a mark, then?

Kl: Why not a comment?

Rb: I would rather have a comment, saying that we're working hard, doing the best we can, than getting a mark.

Cl: But then again, if you think about it, they're not going to accept you in university, if Mr. Mason writes a letter and says we're working hard, and doing, you know.

Kl: [laughs] And here's my letter of reference. [laughter]

Cl: Yeah. I don't have a mark for grade nine, but here's a letter from Mr. Mason.

Rb: Here, he wrote this.

Kl: Yeah, it was real good too.

MrM: Your university's not going to look at your grade nine mark.

Cl: Well say you were teaching us in grade twelve. I don't have a mark for grade twelve math, but here's a letter,

Kl: It says I was working hard! [laughs]

Cl: Yeah, it says I was working hard, you know, it says I was in the top 20 of 100 kids, and, it's like, whoopee, they need concrete marks.

MrM: If you were in grade twelve, do you think that's all I would do for you?

Cl: In grade twelve? No. I think that you are concerned with our future, and you, I think that just out of, just because how the world is, you would have, kind of, I don't know. But I think if I was teaching, you know, I would have felt obligated to give a mark.

Kl: That's what people are looking for.

MrM: It's also in my contract.

Cl: But I'd give it, and I'd say, this may get you into university, but I don't want you to concentrate on this. I want you to concentrate on what my letter said, because this tells you more stuff than just a mark. A mark, 89%, there, wow. But a letter, he did good, but sometimes his homework wasn't complete, and sometimes he talked too loud, or something, you know.

This is the quandary of assessment in mathematics reform. Assessment as feedback is vital to students who are undergoing transitions in their senses of mathematics, of learning, of themselves, and of schooling. Yet assessment as sorting is viewed as vital by the school as an institution, and that means students must attend with vigilance to the sorting function of assessment. Assessment in its form and substance defined for these students what counts as mathematics. Whether teachers "teach for the test" or not, students will learn for the test, as long as the outcomes of the tests (weekly, monthly, annual, or terminal) are so significant. When students must defend their senses of self from the constant sorting-oriented evaluation that they face, they cannot easily use the processes between evaluations as avenues of personal change. Nor can they view the results of those evaluations as feedback fostering change. Their opportunities to learn ontologically and epistemologically are severely impeded by the domination of marks and tests in their perceptions of academic success. School mathematics is indeed determined more by its context than its content, at least from the perspective of these three students.

In this conversation Ruby has occasionally shown a more multi-faceted orientation to the questions of mathematics assessment. Her March 11 homework response, following the algebra tiles test, reflects such an orientation too. The question was "How should you get a mark out of 100 for the test?" Ruby replied, "Again this question is difficult for me to answer. You may consider 8-10 ways to mark me and I may consider different ways. How is it that one mark can reflect all of my efforts, hard work, learning, learning skills, understanding etc. and how could I come up with the same number as you? I can't, sorry!"

Ruby suggests a relativistic orientation (Baxter Magolda, 1992; Belenky, Clinchy, Goldberger, & Tarule, 1986) as well as a stance which rejects numerical summaries of mathematical experiences and achievement. She implies there is a richness in the factors she considers important, but could her use of "etc." imply that she may not have made these factors explicit in her reflections yet? Clearly, students are capable of broader perspectives on issues of what counts in mathematics, and broadening students' perspectives on this issue should count high on the mathematics reform agenda. Discussing assessment issues and sharing assessment decisions with students can be an effective strategy concurrent with instructional reform, and that it is quite likely to help researchers develop their perspectives on the issues and strategies as well. It did for me.

Mathematics reform and student autonomy

The next portion of conversation again begins with my asking the students to look back on a particular time. The two times which are discussed were the subjects of prior chapters which included Kelsey, Colleen, and Ruby directly (Chapters 5 and 7). The students made their own unique sense of those experiences, just as I did in the earlier chapters. One of the exciting elements of this passage for me is the predominance of statements that begin "We wanted" and that express in active voice choices which the students made. Intentionality is clear, and autonomy becomes a clear element as the passage continues.

> MrM: The day that you and I talked before Christmas was probably a number one day for you guys as a group. But I had no idea what was going on in your heads that day. A month later, you got a chance to design your own project, and I think the designing of the project was a more important day than the days you actually did the project.
>
> Cl: Yeah, because it took a lot of thought to figure out what we wanted to do.
>
> Rb: Yeah.
>
> Kl: And how we wanted to get it done.
>
> Cl: Because we still wanted it to be challenging, but we didn't want it to blow us away.
>
> MrM: You also said, you didn't want it to be about answers.
>
> Cl: No.
>
> Kl: We wanted to know how we learned, us. How much I learned, personally. Or how much Ruby learned, or how much Colleen learned.
>
> Cl: And we did. I think we did a very good job on that.

Kl: I think that project was very,

Cl: It was very challenging, and we did a lot of work.

Kl: Exactly.

Rb: It was the sort of thing we had been doing all year long, so to get a question like that and be able to approach it all by ourselves,

Kl: All by our own selves.

Cl: All by ourselves. You know, we asked you the odd question, but you didn't just go up, you didn't just say, read the question, and make an algebra for it, you just kind of said, okay, here's your question. Have fun.

Kl: Have fun. It's like, do what you want to do.

Cl: And if you just get the answer,

Rb: If that's what helps you learn, go for it.

Cl: But if finding five billion different ways to get the answer to one question, if that's what you want to do, then you do that. and that was weird, because we never got to make our own choices before.

Kl: We still don't. This is the only class I get to make choices in.

Cl: Yeah, this is the only class, and it's only when you're here.

MrM: So making choices helps you learn?

Cl: I think so.

Kl: Yes

Rb: Cause if you make the wrong choice, you learn from it. If you make the wrong choice then you still learn from it.

MrM: Do you learn from your wrong choices?

All: Yes. Definitely.

MrM: If you guys had chosen not to work with me?

Cl: Oh, we would have regretted it.

Kl: I think I would have regretted it.

Rb: Well, we probably wouldn't have regretted it till later,

Kl: Not till later on.

Cl: Not until later, because we wouldn't have realized what we were doing.

Rb: But when we saw the Math 30 exam, we'd think oh my God, that's what we were learning in grade nine, why didn't we stick with him.

Cl: Yeah, we would have kicked ourselves so hard.

At the time, I was feeling proud of their sense of value for what we have done together, and I still find this gratifying as I reread it now. I also find the idea that grade nine students believe they make no choices as students quite scary. Somehow, despite their self-reported lack of experience with such possibilities, these students recognized the opportunity to manage aspects of their learning, and they believed they made the most of it.

Near the end of the transcript above, there is a shift which has occurred in previous transcripts: when the talk turned to what counts in learning and studenting, tests became an operational definition of great consequence. However, Ruby referred to the test to make an extraordinary claim: for a test which she hadn't yet seen, and which she knew was likely to be neither predictable or controllable, she believed that what she had learned would prepare her well. Below, as I pursue this thought further, they express that it isn't specifically for the exam that they value what they have done: that was only a convenient (and perhaps habitual) way of saying that a learning experience will pay off. Their way of saying this brings back elements of previous statements made to them by Mrs. Larkin (Chapter 5).

MrM: [laughs] What if you get to the Math 30 exam, and you find out it's just, "show what you know?" .
Cl: Then I'll show what I know, but I'll still have this, tucked back in my mind, ready for use.
MrM: Ready for use? Where are you going to use it if it's not on the Math 30 exam?
Kl: University,
Cl: University, in life, like, problem solving, there's math problem solving, but there's also PROBLEM solving right,
Rb: Like real problem solving.
Cl: Yeah, you're teaching us to go about problems in different ways, and that can apply to anything. That can apply to anything,
Rb: It doesn't just have to be math.
Cl: Whether you're going to have a kid, whether you're going to get married, whether you're going to take this job or that job,
Kl: Whether you're going to get A job. [laughs]
MrM: So we'll call your unknown kid KID colon,
All: Nooo! [laugh]
Kl: Or SOS colon.
MrM: Let me know what you mean by applying this.
Rb: What we're doing here is we're sort of learning how you can get an answer,

like if you had a question,

Kl: By looking at it in different ways.

Rb: Like if you had a question like, You can either do a social essay, or you can do a social report on something, you can look at this thing and say, the advantages to doing this is this, the advantages to doing this is this, instead of just saying, Oh well this will be easier I think I'll do this one.

Cl: Yeah, but now, we'll just think about things, okay, now what are my advantages if I do this, and what are my advantages if I do this?

Kl: And what are the disadvantages?

Rb: And if I do this, I can get information from here, and if I do this, I can get information from here.

Cl: Kind of planning. We're going slower, we're not going to speed through our lives, we're going to take time to think about what we're doing, and probably make some good choices I hope.

Kl: I hope! [laughs]

Despite the appeal of such powerful claims, caution would be important with what these students are saying. Transfer is a huge issue in learning generally, especially in learning to think and learning to learn (Keefe & Walberg, 1992). These students spontaneously claimed that what they had learned had relevance to them in other decision fields, including other learning choices and life choices. Yet this is not the same as evidence of transfer: although they have predicted the relevance of what they learned, they have not actually used what they have learned in future decisions. As he conversation is rejoined, I help the conversation return to the cognitive aspects of what they said.

MrM: Even when I'm not around?

Kl: I think what I've learned here, I'll always remember.

Rb: Yeah.

Cl: Always. I hope.

Kl: I hope, yeah. In this class I've become more open-minded.

MrM: Remember or memorize?

Cl: Remember.

Kl: Remember. I can't memorize what you said.

Cl: I can't memorize this. You can't memorize how to make a decision. You remember,

Kl: On what you did the previous time, and live through that.

Cl: Yeah, not the decision you eventually make, but how you went about making it.

Rb: You can memorize formulas to help you, but you can't memorize ways to live.

Kl: You can memorize formulas till your eyes are blue, but maybe they won't apply.

MrM: Then why do we spend so much time teaching you guys formulas?

Rb: Because certain things they apply. If you want to find the area, instead of saying I can take this and I can take this and I can take this,

Kl: You could just do it.

Rb: You could say, hey, I have to do this and this and this, and I've got my area. But for a lot of things, it's not that cut and dry, and you can't just think, isn't there some of kind of formula I can do here? You have to think, I can do it this way, or I can do it this way, and then I'll see which one's easier for me, or which one's better for me.

This answers a question posed earlier in this chapter: should teachers equate making learning simple for our students with helping them learn? Ruby's answer is straightforward, but not simple: there are simple ways to answer many of life's questions, but there are no simple answers for other questions. "It's not that cut and dry." I like her answer. All three students have held on to a sense of the validity of their previous ways of learning -- the listening, the memorizing -- as they have developed a sense of the validity of more autonomous ways of learning in more ambiguous action space. That they have not rejected what worked for them in previous contexts suggests that what they developed with me were not replacement epistemologies and ontologies but more complex stances inclusive of their previous experiences, in terms of what counts as knowing and in terms of how their sense of self and sense of the role of student aligned. Again, here is the underlying theme of this chapter: these educational successes are best viewed when the perspective incorporates the realities of school math, with its tests and marks and intentions to sort students as well as to teach them.

Ruby, Kelsey, and Colleen have shown in this chapter that their view of and interaction with school mathematics was dominated by its tests and grading. Are we going to throw students against an impassive ranking structure, year after year, as the wind throws waves against the shore (Paquette, 1995)? This study has not pursued evidence of the relative justice or injustice of marks-driven curricula. Nor has it studied the wisdom of the school system using mathematics as its primary critical filter. Both of these elements would need to be a part of a full answer to questions about reorienting the assessment in school mathematics. This study has provided evidence in two areas relevant to such questions, one which suggests the importance of the questions and the other which suggests hope concerning possible answers. School mathematics as many good students perceive it guides them to view their role as to perform and conform rather than to learn. They do not perceive mathematics as a subject where they can risk the development of broader orientations which could help them learn more completely. On the other hand, mathematics can be taught in ways where students can come to define their success increasingly in terms of their understanding of the mathematical concepts through inquiry, interaction, and reflection. Furthermore that success can include (and likely depends upon) the students' development of deliberate orientations to learn mathematics in more personal, interpersonal, and autonomous ways.

School mathematics need not be the relatively impervious body of knowledge it was when it became a very effective subject for sorting students. Students can be helped to change the ways in which they view mathematics, learning, their roles as students, and even their relationship with the structures of the school system, as suggested by this research. The implications are that mathematics can be accessible to many more people, a subject more useful and more meaningful to many more students and a pathway to a sense of competence for many more learners. There are better goals to be accomplished through school mathematics now than the sorting of students.

Mathematics education reform in its fullest context: a summary

This study has been an exercise in the pursuit of those better goals. Beginning within a constructivist sense of how people come to understand, curriculum was designed which was built upon active and interactive processes of inquiry. As students encountered processes which were different from their prior experiences within school mathematics, support for their questioning of the new processes and their prior experiences (reflecting) was offered through three interactive processes: peer discourse with and without a teacher's guidance, a daily invitation to write to the teacher, and in-class conversations which affected both instructional and assessment processes. Within this support framework, students were encouraged to question the assumptions they had about how learning happens and about what counts as knowledge/understanding (epistemology), about their sense of self and their own becoming as students and learners (ontology), and about their relationships with the authority structures within school. Through the students' actions in each of those realms of inquiry and their interactive reflections on their actions, the students constructed richer understandings.

In previous chapters, a number of statements have been made which respond to the purposes of the study as made explicit in Chapter 2. They are restated below in a list, without the richness of their original development. The purpose of the list is simply to highlight particular features of the complex relationships (among the particular people, and among the specifiable components of the teaching and learning processes) which have been portrayed in this study as central to the processes and outcomes of this research. Each statement could be considered as a research conclusion within the context of the data interpretation which sponsored it, attributable to a particular chapter in this document. However, just as the mathematical and non-mathematical elements of the learners' experiences intertwined and developed interactively over the full period of the inquiry process, these conclusions are best taken as interdependent strands woven within the whole fabric of described events. Although the statements are made without direct reference to the contextual elements which were part of their development in previous chapters, there is no intention here to claim general applicability for these conclusions without consideration of the contextual and personal factors which would determine their validity in other situations.

--Constructivism as a learning theory can successfully guide instructional decision-making; in particular, activities can be designed which provide an experience base to sponsor students' construction of viable understandings of the forms and arithmetic of algebra.

--Reform of instruction should not (and probably cannot) proceed without anticipating, recognizing, and supporting students' need to reorient themselves significantly to the reforms.

--Students' reorientations are educational events, to be anticipated and planned for and supported during instruction. A constructivist orientation to thinking about how students might construct new orientations can support and guide these educational decisions as well. In particular, students need opportunities to succeed with reform-oriented instructional activity, and they need guided opportunities to converse and reflect on their successes.

--Multiple layers of discourse can provide support for students who risk challenging their prior conceptions of mathematics and how to learn it. Such discourse can include small-group conversations with and without the teacher and full-class conversations, anchored by unmarked written exchanges between teacher and students. In these forms of discourse, the students can be encouraged to switch their reflective focus from doing mathematics to talking about learning mathematics, and from talking about learning mathematics to talking about what the learning processes mean to them.

--The students' sense of vulnerability to the sorting functions which have been attached to mathematics can itself inform the pursuit of mathematics reform and help to shape it. To achieve this, instructional reform should provide support for students as they address their senses of risk and of opportunity, and question the validity of the sources of those risks. In terms of their impact on students, when the students have come to understand their relationship to those sources of risk and opportunity differently, for the students the sources themselves are no longer the same. These outcomes are fostered further when the students can exercise influence on the instructional and assessment structures of the classroom through the relatively autonomous choices they make, especially if they can perceive and reflect upon those choices and the influences which the choices achieved.

--Reform-oriented research can be situated in schools and be predicated on success for the students, inclusive of both mathematical understanding and academic success. Such research needs to be multi-faceted in its design, both in terms of the number of elements to be manipulated and the number of elements to be observed. Although there may well be powerful generalizations available from within this constellation of elements, instructional actions and data gathering should be conceptualized recognizing the deeply personal, interpersonal, and context-specific nature of learning and of students' relationships with school structures.

I have called the above statements conclusions because for me the experiences I have had within the context I have described are completing convincing for me. This is different from saying that these are conclusions in the classic research sense. I have not presented evidence which compares what these students understand to what other students understand or to what they would have understood without changes in their instruction. Nor have I framed the research in such a way that demonstrates that these outcomes and findings are transferable to other students, other teachers, and other contexts. These are the beliefs which led me to shape the research to be context-dependent rather than to pursue a claim which is demonstrably generalizable: that learning is a deeply personal (and interpersonal) act; that teaching is deeply dependent for its character and potential on the individuals involved and their interactions; that the essence of education and schooling is the dynamic relationships determined within each classroom by the constant and ongoing construction of relationships, structures, and meanings.

Rather than thinking of this research as being conclusive for its readers, I would recommend considering it exploratory in the classic sense of that word. Are the choices and considerations "on the right track?" That may not be best framing of the question--I have not laid down a railway for you to follow, nor even surveyed a route. I have, however, journeyed where others may choose to go, and I hope I have identified salient features of the landscape in such a way as to make others' navigating easier. I hope also that I have identified positive possibilities of the landscape I explored and of the procedures of the exploring, so that the journeys of others may be similarly fruitful. If there are routes to be delineated and surveying to be done, that remains for others.

In Chapter 2, I suggested that this document could be considered a story, a purposeful portrayal of a sequence of events. To the extent that this writing is a story, what kind of a story is it? If it needed to be categorized according to its style, I would make up the category, *story of research*. However, a more important answer would be based on my intent in telling the story: I consider it and have written it as a *success story*. It is about the success of an ambitious instructional approach in terms of mathematical understanding. It is about the success of that instructional approach in terms of students' understandings of themselves and their ways of learning. It is about the success of a research design which attempted to accept the embeddedness of pedagogy in the powerful (and empowering) context of particular classrooms and the embeddedness of instructional efficacy within pedagogic relationships with particular students. Finally, particularly in the last chapter to come, it is about the success of the researcher as a learner, especially in terms that I neither predicted nor controlled. Research in mathematics education and education generally needs more stories that are less about the barriers to success and more about successes in themselves.

There is one more chapter. No exploratory journey would be significant if the explorer returned unchanged. I describe one last event, a story of my learning about me, and the student (the first student you met in Chapter 1) that provided the opportunity. As I express my thanks to her, it is not for this one event, but for the whole strand of shared experience which I hope is represented through this one event. Also, as I thank her, I thank all those with whom I have shared parts of this journey who were open to my influence and thus were able to influence me.

Chapter 9. Epilogue: A Quiet Learner Teaches.

This chapter returns to a thread left dangling since the first section of transcript which opened Chapter 1. Benazhir returns into view, but the story here is more about me: as teacher-researcher, I faced decisions I had never faced before, which meant that I became the learner at Benazhir's provocation. I learned about who I am and I made progress toward who I want myself to be. I learned about some absolute limits to my professional efficacy. In response to the autobiographical approach of this chapter, the tone of the telling shifts from that of previous chapters. Among its many possible meanings, this story represents a different treatment of issues of the influences of gender and culture within a mathematics classroom. The diversity within this stream of academic voices will offer various starting points for framing the story to come.

> The pretense operating in many schools is that teachers should treat all students the same, although numerous studies on teacher expectations have shown that race, class, and gender have considerable influence over the assumptions, conscious and unconscious, that teachers hold toward students (Noguera, 1995, p. 203).

> One reason why the performance of females is not sustained through the secondary school years is because of their compliance, and consequent dependency, on the authority of the teacher (Burton, 1989, p. 17).

> Culture is no longer viewed as static, one-dimensional, and uncontested, but as having multiple layers. This significant reconceptualization of multiculturalism interrogates the creation of difference within the context of history, culture, power, and ideology (Schwartz, 1995, p. 636).

> We do not think of race and gender oppression in additive terms, an implication of phrases such as *double* and *triple jeopardy*. Rather, race, class, and gender are part of the whole fabric of experience for all groups, not just women and people of color (Anderson, & Collins, 1992, p. xii).

> The color-blind assumption can infer that student difference and defect are synonymous. It may result in missing opportunities to build on the lived experiences of many students. What appears fair may only exacerbate inequities (Tate, 1995, p. 345).

> Speaking the language of critical pedagogy is neither necessary nor sufficient for building a diverse democratic community within the schools. This is the postmodernist trap that confuses a change in language with a change in the world (Seixas, 1995, p. 436).

As a tall and able-bodied, white, male, English-first-language many-generation Canadian, I had tried hard throughout my career to respond to students fairly and equitably. In doing so I felt that the significant authority so readily granted to me as a teacher was seldom my ally in terms of the professional goals I had set. Too often the instructional opportunities that developed within pedagogic relationships with students were limited by the authority invested in me because of my position. In a less defined way I felt similarly about authority that fell to me because of my membership in various groups to which authority has traditionally been granted. I had considered myself to be appropriately positioned as an educator by trying not to activate or respond to factors of gender and ethnicity and cultural background.

This stance did not align well with my own growing consciousness of the significance of every learner's racial and gender identity in education. Nor did it align with educational readings I felt compelled to agree with that suggested that identity was necessarily gendered and cultured (Anyon, 1994; Apple, 1992; Briskin, 1994; Code, 1991; Duran, 1991; Fine, 1989; Mansfield & Kehoe, 1994; Marshall, 1994; Stotsky, 1995). Yet I had not found myself in a circumstance which challenged my pedagogic stance of assumed neutrality. Benazhir provided that circumstance. The particular incident occurred two days before the class from which Chapter 1's transcripts were selected. It was not a significant event for Benazhir, and its meaning to me did not develop to an expressed form until weeks later.

The circumstance surrounded my providing a letter to accompany each student's second report card, as I had done for the first report card in November. Near the end of a lesson I asked each group to discuss what topics the letter should prioritize and then the question became their homework for the day.

In general Benazhir's responses to the homework questions were brief. She did not perceive her own learning as something to learn about, and as a result she saw no reason to participate extensively in something that wasn't worth marks and wasn't covered on tests. This invitation, however, was one Benazhir accepted readily--it involved her marks and her parents and that mattered a lot. Part of what she wrote was, "You should say how I stress that a lot of my mark be based on the effort and work I put into my project. Don't put a lot of emphasis on homework because I found it quite annoying at times, when you asked insignificant questions or things that didn't relate to the topics or subjects of our classes." When teaching received knowers (Belenky, Clinchy, Goldberger, & Tarule, 1986; Baxter Magolda, 1992), it is worth knowing that they are unlikely to consider their way of knowing as being worth talking about or learning about, let alone changing; to the extent that they care about these things, they would expect an expert to be the source of their knowledge about themselves in this regard.

That characterization described Benazhir's curiosity well. In the same homework reply, she suggested, "I have no idea what my learning style is, and would like for you to say what mine is (I'm quite interested in finding out). It's a good idea for you to say what you are trying to do with us and where we are going in the letter home." In the same homework response in which she dismissed the structured reflective activity, Benazhir asked to be told by me how she learns! I do not recall whether I reacted eagerly to the challenge of a direct request to explain something I wanted the student to construct herself, or whether I was more miffed by her blunt dismissal of the homework process.

Two days later, I provided each student with a rough draft of my letter to their parents. Each student's letter had common sections describing the instructional project and the class activities for that term, and customized sections describing the nature of each student's progress during those classes. The students were encouraged to comment and give me feedback before the letter was finalized, in part because their involvement made the process more accurate and meaningful for them, and in part because providing input might involve the students metacognitively in deciding what was worth reporting about their own learning processes. Among the class members the response to the invitation varied as much as it did for any of the homework questions.

One particular point of discussion was why each student's "mark" from me only estimated how well their group was doing, compared hypothetically to 100 similar groups. For example, Benazhir, Lorna, and Rose were each told in their letter that their group was performing in ways that suggested they would likely be in the top 20 of 100 groups at their grade level. Both Rose and Benazhir wished for more exact percentages but they agreed as Ruby, Kelsey, and Colleen did in Chapter 7 that the kinds of tests that could generate such scores wouldn't really be appropriate for the kind of learning they had done.

Few students made suggestions for changing the draft. Generally the contents were not surprises to them and they already had had significant input into its formation. They had seen their parents react positively to the previous letter sent home in November. Benazhir was not comfortable, however, as we might expect from the opening snippet of classroom conversation included in Chapter 1. "I'm so unlucky. I'm getting my report card tomorrow. And I'm getting the lowest marks in my life tomorrow. . I'm getting like the lowest marks in my life tomorrow. I'm so unlucky."

Two topics in the letter bothered Benazhir. One was her absences, which had averaged more than a day per week since the beginning of the year, although these absences didn't fall on days with research instruction any more or less than other days. The previous week I had asked Mrs. Larkin about all of Benazhir's absences but she had replied that Benazhir's absences were a non-event both to her and her other teachers. There was always a note from Benazhir's mother, acknowledging the absence as having approved by her. Following any absence Benazhir was scrupulous in catching up on any missed work and notes, although she had twice complained to me that the kind of stuff she missed in my classes was hard to catch up on. Like the teachers, and for the same reasons, I gave the absences no further thought. However, I mentioned them twice in the letter home. Once was simply a general comment, "Benazhir missed a significant number of periods." However, one other pattern had emerged, which I had chosen to address. Although within the research project it was always acceptable for any student not to have any reply to previous homework, Benazhir had said to me more than once that the reason had been that she had been away the previous day. This was given as a reason whether the previous day's math class had been a research period or not. In describing Benazhir's limited homework responses, I stated, "Benazhir's efforts in this area have also suffered through her absences." It is understandable that Benazhir didn't feel comfortable with this.

Benazhir wrote in the margin of the rough draft she returned to me, "I don't want this in here because my parents know I've missed quite a few classes and don't need to be reminded or told about it in here. And you make it sound like I'm not really doing good." Benazhir was right in one way: I did not want to imply inadequacy of performance to her parents in regard to the homework. Homework wasn't supposed to be marked even indirectly like this, but my coercive habits of fifteen years in the classroom kept sneaking in even when I didn't want them, and this seemed in retrospect to be one of those times. The comment was a pushy attempt to get Benazhir to try more often with the reflective homework and that wasn't how I wanted to have students approach that opportunity. Yet I did feel the desire to mention that the absences created a discontinuity in Benazhir's program, so I improved the wording and showed it to Benazhir, and she didn't object, although she reiterated that she didn't think it should even get mentioned. I told her I understood her feeling that way but that I did not agree with her, and she agreed that we might disagree that way. Benazhir had seldom had chances to negotiate the content of evaluations and reports to parents, and was quite prepared to accept my authority in the task. In fact, I was pleased at the time with her spunkiness about the concern, and hopeful that it might be indicative of emerging autonomy regarding what counts in her learning.

It was Benazhir's other sentence about the rough draft that triggered a watershed event for me. The sentence to which Benazhir reacted was one which began with a standard form for all students, and closed with issues and elements of significance from the students' "homework" writing with me. In Benazhir's report, it said, "I would encourage you to ask Benazhir about any aspect of the instruction. Areas which perhaps you can discuss with her are whether good questions for learning math should be precise and describe exactly what students need to do, and whether answers are more important to learning math than the processes which get answers."

Benazhir wrote on her draft before returning it to me, "We DON'T discuss math at home." I didn't know what she meant, and no other student had reacted except favorably to this use of themes from their personal homework writing in the letter home. When I followed up this comment with her the following day, simply because I could not understand Benazhir's concern, she was somewhat reluctant to explain in detail, but she did say two things. "My father doesn't talk to his daughters about ANYTHING. And my mom can't encourage me at all, except in cooking and looking after my sister and stuff, or my dad will get mad." I was perplexed. I had anticipated that her concern had been in the themes I had suggested, but it was actually about the possibility of discussing math and learning at home.

I mentioned this to Mrs. Larkin, hoping for some insight. She thought quietly a moment and then she said, "Of course. And that explains the absences. I knew that the family was Islamic, but Benazhir's father must be very devout." If this is so, and if this is relevant, then Benazhir wasn't even supposed to succeed in school. She was supposed to be preparing herself to be a wife and mother, and her absences more often than not were likely in relation to her preparation for that certain future (Gerami, 1996).

This explanation made many elements so much more understandable, such as Benazhir's considerable concern about what reports to parents said. It provides meaning to the limits of her conversation with friends at school. Lorna and Rose were her best friends in school, but Benazhir didn't share certain aspects of her home life with them. It is suddenly relevant that at the end of transcript which opens Chapter 1, Benazhir said, after introducing the topic of little sisters and parents' reactions to marks,

Bz: I'm a very busy person out of school.
Lr: I know.
Bz: You don't. I hope you never find out either.

More important, this conceptualization could explain in part Benazhir's fervent hope that memorizing and doing the assigned work would be enough for her to continue to excel. Missing more than 20% of the time, she almost had to hope that class-time experiences were not terribly educational, for otherwise she could not catch up by doing her work on her own. Even when she found appealing the image of the learning opportunity as being experiential and social, and even when she found appealing the richer more personal image of understanding being offered, Benazhir could not buy in completely without creating an unresolvable conflict with relation to her desire to succeed both at school and with her family. Rather, she had to hope that the essence of each class she missed could be preserved and absorbed from a classmate's notes, and that the value of the activities could be recaptured by catching up on the work missed. This wouldn't resolve the conflict inherent in her goal of excelling in school and her family's goal for her, but it would not demand that the resolution be immediate either.

Yet, we did not inquire further or follow up on this line of thought. At the time (and even now) we felt that our non-intervention was clearly the only valid response at the time. First, the academic literature on the influence of ethnicity on learning tends to assume that ethnicity and school achievement are aligned compatibly, not oppositionally. Deyhle provides a typical example, "The strength of one's cultural identity is a vital factor in the expressive responses to the schooling experience" (Deyhle, 1995, p. 408). To take any action would mean proceeding without any prior knowledge of what action would be appropriate.

Second, even to confirm or reject our simple hypothesis would mean increased involvement in an area that Benazhir had kept to herself. We knew she would have considered it a significant invasion of her privacy for any school agent to get involved. If either Mrs. Larkin or I were to do so, it would have been a violation of her particular trust in us: especially with the extra lines of communication which the tape recorders and 'homework' responses represented and the much broader range of our in-class conversations, we knew without doubt to what extent she was open to our knowing more about this. She had not at any point invited action from us in relation to this issue, and to engage in further inquiry would be to engage in action. "Sometimes women maintain silence for survival. Sometimes women are silent to maintain privileges. It is important to be clear which is which: ...silence as safety, silence as control, silence as communication, silence as political act, silence as hearing, silence as withdrawal. ... silence as a tool of power ... silence as a tool of resistance ... silence as complicity" (Beagan, 1996, p. 113). Further, even if our hypothesis was confirmed, there was little possibility of our taking any helpful meaningful action. We could only continue to do our best as teachers dealing only with those aspects of Benazhir's circumstances which she declared relevant or which she declared accessible by sharing or bringing them forward to us. This element which was so significant to Benazhir's sense of herself as a student and her sense of what learning mathematics should be like was closed to our inquiry and our influence.

This story could be construed as evidence that getting involved with students' ontology and epistemology may not be fruitful, and that is not my intent at all. Rather, I have two purposes in telling this story. One, it is further evidence of the interrelatedness of personal and social and institutional elements of teaching and learning. Benazhir's choices of how to learn math, of how to succeed at as a student, were inter-related with her parents' view of what was culturally appropriate for her, and that was very much relevant to the issues of how we her teachers could interact with her. The fabric of a student's sense of herself does not respond well to unweaving, and actions in relation to any of the strands will affect and be mediated by the whole tapestry. That applies to factors which the students bring with them, not just the factors which the teacher or researcher declares relevant.

Second, the story captures for me an essential aspect of the pedagogic problematic--the inherent limits of any interpersonal enterprise. The methods I am describing, as effective as I found them to be, were not and could not and should not be so effective as to override all the other elements of the reality in which I taught and learned. Benazhir provided a challenge which I could not address or decide how to address. The limits of teachers' authority and parental authority did not allow the pursuit of an accommodation that would assist Benazhir in finding her desired autonomy in her relationships with school and home. To understand one's students more completely is an essential aspect of teaching students more completely. However, understands one's students more completely does not always mean one understands more completely how to teach them.

There was one complex action I could initiate, however, even if it had no effect on Benazhir's conflict. This event provided incontrovertable evidence that my stance of selective blindness regarding gender, ethnic background, race, and religion was pedagogically immoral. A denial of the centrality of categorical differences in others' views of the world limits one's understanding of their world and that limits one's opportunity to be pedagogically effective for them. This new awareness had one central element: how should I perceive myself in this world of classrooms where so much authority accrued to me, wanted or not? How naively arrogant of me it was to pretend I was not at all times a gendered and otherwise categorized person when I was teaching! How arrogantly naive it was to pretend that selective blindness was ideal modeling of an equitable and just perspective of others in one's world! Ethnic blindness was no longer a comfortable or a safe alternative for me as a pedagogic stance.

Rose, Benazhir and Lorna's group-mate in Chapter 1, provided a less problematic example of how ethnic blindness can distort a teacher's view, my view. To my ears Rose spoke unaccented and unrestricted English in this her fifth year in Canada, although her word endings could have been categorized as slightly clipped. Can we fully address her relative quietness in regard to her role as a learner of mathematics, however, without wondering what her preferred level of verbal involvement might have been over those five years, while she was learning such flawless verbal English? Voice (literal and metaphoric) is clearly an important element in student learning and in students' sense of self (Gilligan, 1982; Belenky, Clinchy, Goldberger, & Tarule, 1986; Baxter Magolda, 1992; Chickering & Reisser, 1993; Gutierrez, Rymes, & Larson 1995). It was not just incorrect for me to ignore her history to maintain my sense of my own ethnic centrality. It was pedagogically debilitating, if it led to simplistic understandings of how Rose learned.

Perhaps even now I should be more critical of my willingness to characterize Rose's confidence in her spoken English, as I did in the previous paragraph. How then should I now look back on my interactions with her? Here is a short sample from Rose's homework. Was it appropriate for me to position myself as if I were a neutral element in questions of ethnicity and language? On this day (November 19), mathematics was during the last period of the day. The learning awareness question I asked at the end of the period was, "Describe a learning moment from a different class today." Rose's response said, "The learning moment was in L. A. I learned a few metaphors that I didn't know before like, 'He is a rough diamond' and 'Their jokes broke the ice at the party.'" I asked the language arts teacher and some of the other students from that class about the lesson: in their opinion, the lesson had been about topic sentences. What would you have written back to Rose that day, which would respect her having written to you? My response only dealt with the humorous possibilities of taking common idioms literally.

This line of thought suggests that it is risky to take action in a world rich with a sense of difference. Yet, I do not wish to imply that we should avoid that risk. "Exposure to difference encourages breadth of attention, a way of seeing that underlies ways of continuing to learn as an adult, for every opening to different cultural traditions is a rehearsal for dealing constructively with inner or outer change" (Bateson, 1994, p. 168). In fact, it could be risk avoidance that motivates us when we choose to be ethnically blind to the richness of the differences others see. In its way, it is suggestive of the problematic nature of any pedagogic act: it cannot take place within a complete understanding, for it takes place in an infinitely complex web of relevant elements. We cannot know all of those elements, not only because of their plenitude, but because of their nature.

Within a knowing relationship with Benazhir inclusive of her family's religious heritage or with Rose inclusive of her linguistic selfconsciousness, I could only act in an interpersonal way within the personal limits which my particular sense of myself defined. Coming to know more about Benazhir or Rose did not activate those limits. Rather, it activated my awareness of those limits. In the awareness of limits is the opportunity at least to proceed effectively within them, or perhaps even to stretch the limits to some degree in pursuit of pedagogic effectiveness.

By the end of the research strand, I had started to see myself as a particular other for the students, as someone not able to set aside his gender or any other category that was a part of how others saw me. Thus, in the closure conversations, as students discussed with me which high schools they were going to attend, I could actively listen to their claims that this was likely the last year that they would have close friends from other racial groups, as they knew people were expected to eat lunch with and "hang with" people of their own ethnic identity (Deegan, 1996). As they willingly gave details about their perceptions of this, it mattered to me that they were expressing all this, not only to a teacher/researcher they trusted, but to a particular white male English-first-language person they trusted. More important to me, I knew how wrong it would be for me to assume I understood well what they were saying. What each student was saying was about and from a sense of self as a marginalized other. At the time, I simply accepted it as an opportunity for them to teach me what their ethnicity meant to them within the context of their upcoming high school life. In such a frame a mathematics teacher's involvement with students' group dynamics would necessarily be more complex, and would in part be dependent on his/her own singular ethnicity and sense of that ethnicity. A sensitivity to the significance of ethnicity in the students' lives would help to make some educational decisions even while imposing significant limits on the understandings and the actions that the teacher could pursue.

I have suggested earlier that students' sense of themselves as students is deeply linked with their sense of themselves generally, and this must be said of me as a learner too. The changes which I have described in how I saw myself as a teacher/researcher ethnic and gendered in an ethnic and gendered world were also changes in how I saw myself as a person. When I returned to my university offices, enthused and eager to talk about my research with other graduate students, did I recognize that I was Oh-so-gendered, Oh-so-Canadian, to them? Did I pretend to understand how different it must be for my fellow graduate students from southern Africa to leave their families for two years than it was for me to be in a different city from my family for two weeks at a time? How well can I expect myself to have actively attended to the differences in context which their culture and gender added to the mix, differences that stand between what they say and what I understand, along with the distance and time and general personal elements I had readily acknowledged? Nor is this lesson which Benazhir taught me meaningful only in my past. Retrospectives and introspection blend, I hope, as I continue to make sense of myself differently in the directions Benazhir pointed out. Now, as I grapple with issues of First Nations access to our teacher education program and contribute to our community's Women Do Math outreach program, how much of my blindness remains for me to discover?

Thank you, Benazhir, and each and every other of the participants in my research. You ensured that my research was in part about me and for me, in relationship with you. I hope that the honesty I discovered, to learn about my learning and teaching as I invited you to learn about your learning, made a difference to you as you shaped your understanding of the experiences we shared.

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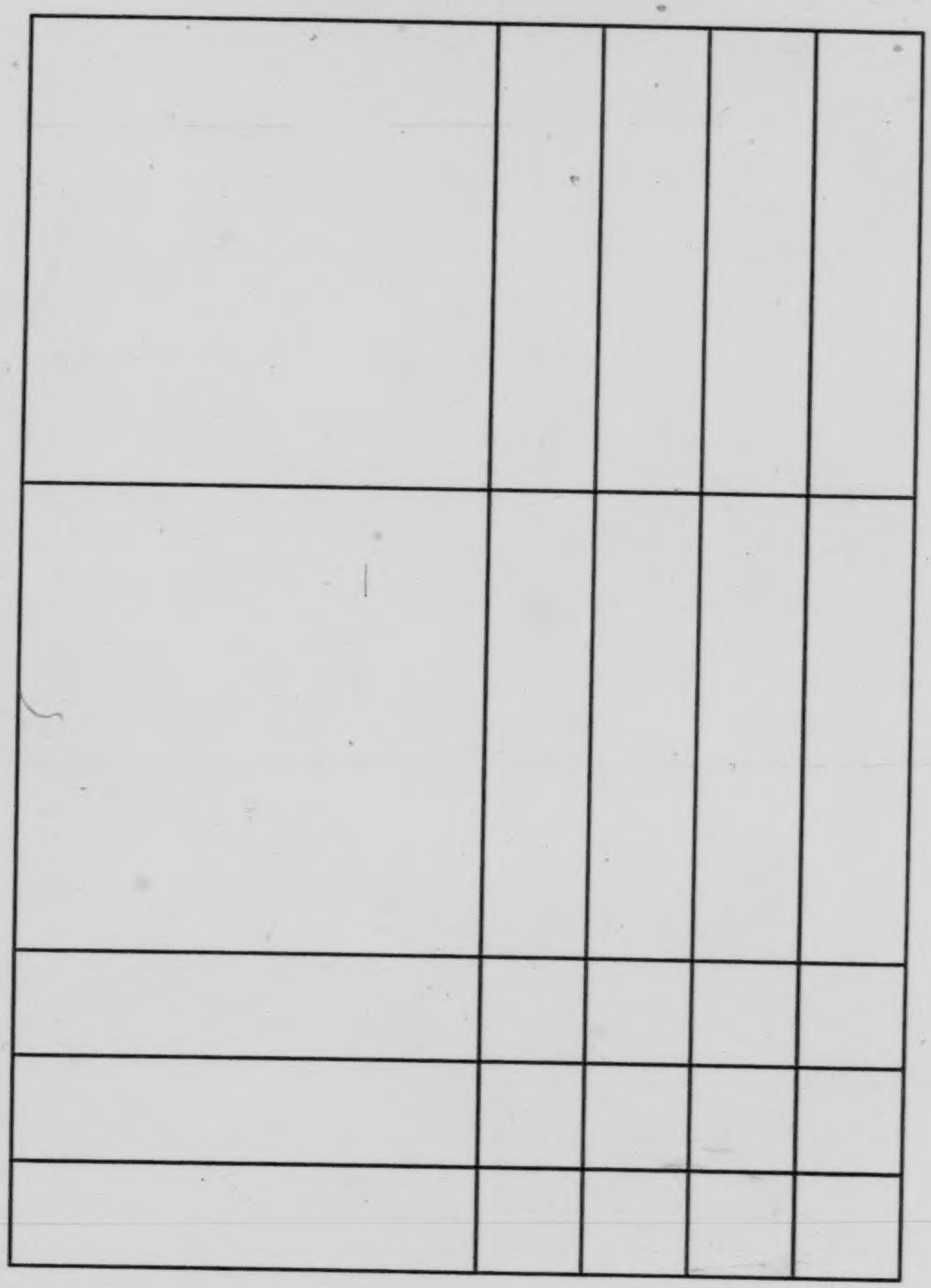
Appendix A. The algebra tiles kit



Appendix B. Algebra tiles sheets

Sheet 1 **Algebra Tiles** name____________________

1. Draw a sketch showing how these polynomials form rectangles. Give both length and width.

a) $2x + 6 =$ b) $x^2 + 4x =$ c) $4x + 2 =$

2. Write in factored form.

a) $2x + 2 =$ f) $x^2 + x =$
b) $3x + 6 =$ g) $x^2 + 2x =$
c) $4x + 6 =$ h) $x^2 + 4x =$
d) $6x + 4 =$ i) $2x^2 =$
e) $6x + 9 =$ j) $2x^2 + x =$

3. [On the back] Show some thinking about parts 2d and 2e above. How about 2i and 2j?

4. Sketch two different rectangles for each, with different lengths and widths.

a) $6x + 6 =$ b) $2x^2 + 4x =$

5. Give a different answer when a question is repeated.

a) $4x + 8 =$ f) $3x + 6 =$
b) $4x + 8 =$ g) $2x^2 + 6x =$
c) $4x + 6 =$ h) $2x^2 + 6x =$
d) $4x + 6 =$ i) $2x^2 + 2x =$
e) $3x + 6 =$ j) $2x^2 + 2x =$

6. On the back, attempt to find a question that can have three or four different rectangles.

Sheet 2 **Algebra Tiles** name____________________

1. Draw a sketch showing how these polynomials form rectangles. Give their length and width.

a) $x^2 + 2x + 1 =$ b) $x^2 + 3x + 2 =$ c) $x^2 + 4x + 3 =$

2. Write in factored form. If the polynomial does not make a rectangle, write "prime."

a) $x^2 + 4x + 4 =$
b) $x^2 + 3x + 3 =$
c) $x^2 + 5x + 3 =$
d) $x^2 + 5x + 4 =$
e) $x^2 + 5x + 6 =$
f) $x^2 + 6x + 8 =$
g) $x^2 + 6x + 9 =$
h) $x^2 + 6x + 10 =$
i) $x^2 + 7x + 10 =$
j) $x^2 + 11x + 10 =$

3. [Back] Compare your answers in question 1 to a partner. Does your partner have a different-looking rectangle with the same length and width? How is that possible?

4. Draw a sketch.

a) $2x^2 + 5x + 2 =$ b) $2x^2 + 3x + 1 =$

5. Factor.

a) $2x^2 + 6x + 4 =$
b) $2x^2 + 7x + 6 =$
c) $2x^2 + 7x + 3 =$
d) $2x^2 + 11x + 5 =$
e) $2x^2 + 11x + 15 =$

6. On the back, do some grid multiplying with some of your answers. Explain.

Sheet 3 **Algebra Tiles** name__________________

1. Tell how many strips you need, to make the polynomial composite (makes a rectangle). Factor.

a) $x^2 +$ ___$x + 1 =$
b) $x^2 +$ ___$x + 2 =$
c) $x^2 +$ ___$x + 3 =$
d) $x^2 +$ ___$x + 4 =$
e) $x^2 +$ ___$x + 4 =$
f) $x^2 +$ ___$x + 6 =$
g) $x^2 +$ ___$x + 6 =$
h) $x^2 +$ ___$x + 9 =$
i) $x^2 +$ ___$x + 9 =$
j) $x^2 +$ ___$x + 10 =$

2. [Back] Some arrangements are simpler than others for a particular rectangle. Explain.

3. Tell how many units you need, so that the polynomial is composite. Factor.

a) $x^2 + 1x +$ ___ $=$
b) $x^2 + 2x +$ ___ $=$
c) $x^2 + 2x +$ ___ $=$
d) $x^2 + 3x +$ ___ $=$
e) $x^2 + 3x +$ ___ $=$
f) $x^2 + 6x +$ ___ $=$
g) $x^2 + 6x +$ ___ $=$
h) $x^2 + 6x +$ ___ $=$
i) $x^2 + 10x +$ ___ $=$
j) $x^2 + 10x +$ ___ $=$

4. On the back, show some good thinking.

5. Tell how many units you need, so that the polynomial is composite. Factor.
a) $2x^2 + 5x +$ ___ $=$
b) $2x^2 + 5x +$ ___ $=$

Sheet 4 **Algebra Tiles** name____________________

1. Sketch these rectangles. What do the rectangles share? What part of the question causes this?

a) $(x + 3)(x + 3) =$ b) $(x + 2)(x + 4) =$ c) $(x + 5)(x + 1) =$

2. Sketch these rectangles. What do the rectangles share? What part of the question causes this?

a) $(x + 3)(x + 3) =$ b) $(x + 2)(x + 4) =$ c) $(x + 5)(x + 1) =$

3. Sketch these rectangles. What do the rectangles share? What part of the question causes this?

a) $x^2 + 2x + 1 =$ b) $x^2 + 6x + 9 =$ c) $x^2 + 8x + 16 =$

4. Sketch these rectangles. What do the rectangles share? What part of the question causes this?

a) $x(x + 4) =$ b) $2x(x + 2) =$ c) $4x(x + 1) =$

5. Show some of your thinking on the back.

Sheet 5 **Algebra Tiles** name____________________

1. My favorite rectangle is $x^2 + 4x + 4$. Sketch three different arrangements of it.

2. If one of the rectangles above was alive, it would grow a little, like the trunk of a tree, or city. Show what $x^2 + 4x + 4$ might look like next year, or even the year after that. Last year?

3. Do you see some patterns worth describing?

4. [Back] Show a growth pattern for a different rectangle.

Appendix C. A brief description of the pattern inquiry contexts

Handshakes: Some people are in a room. They all shake hands with each other. How many handshakes are there?

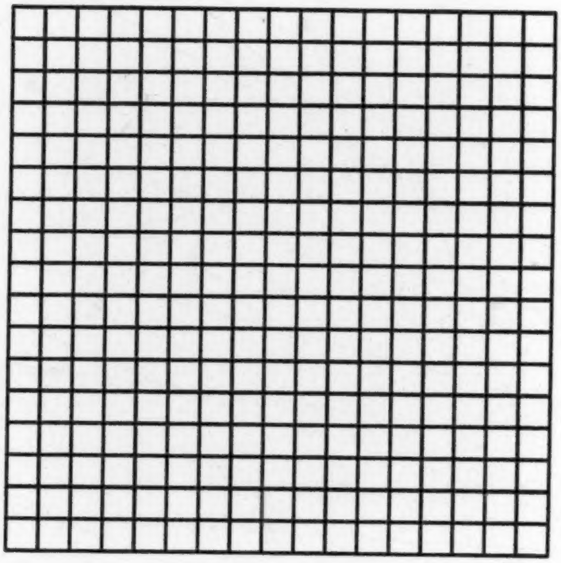
Passways: At a round table conference of Pacific Rim countries, Malaysia was sitting next to Canada, and the two could whisper back and forth whenever they wished. The Japanese delegate complained, "I want to be able to whisper with Canada too. Why did I have to sit between the US and Australia?" Soon everyone wanted to be able to whisper with everyone. The telephone company was called, and rigged private lines from each delegate to every other delegate at the round table, except for each delegate's two neighbors, of course. Each private line was basically two tin cans with a waxed string in between, but everyone was happy. How many lines were there?

Odd Jobs: A university student came to my door and asked me if I was willing to pay her to do odd jobs around the house. I agreed, but I wasn't sure how good her work would be, so I started her small, and gradually increased the size of the job (and her pay for it). I paid her $1 for the first job, $3 for the next, $5 for the next, then $7, and so on. (When she asked me why, I pointed out to her that they were *odd jobs*.) How much money did the student make?

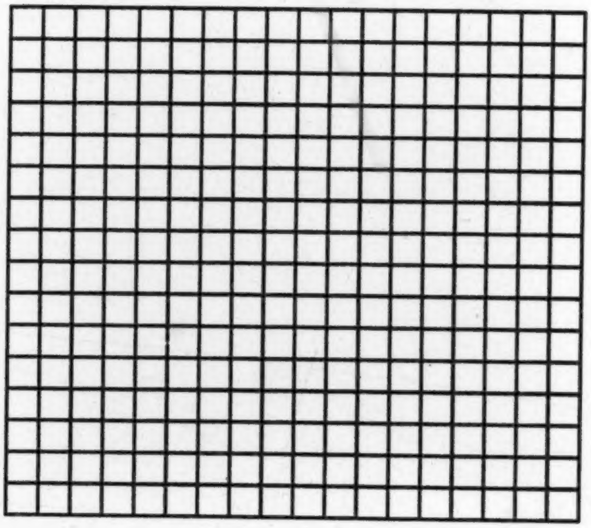
Painted Cube: A cube is made out of 1 cm mini-cubes. Imagine that it is dipped in paint. After the paint dries, the cube is broken down to its component mini-cubes. Some of the mini-cubes have different numbers of sides painted and unpainted. Tell me about this.

Appendix D. A rectangles family

The Parent Rectangle (or Original Square) is 17 by 17; Rectangle A is 16 by 18; Rectangle B should be 16 by 16; Rectangle C is 15 by 15; they can be made from a copy of this page if desired, with the two rectangles cut from the two rectangles provided. Rectangles D-F are not included here.



Parent Rectangle



Rectangle A

Appendix E. A record sheet for exploring the rectangles families

A record sheet for exploring the Rectangle Families page 1 of 2

1. Say something numerical about your Parent Rectangle.

2. Say the same thing about another group's Parent Rectangle.

3. Say the same thing about all Parent Rectangles, using a variable for the length of a side and another variable for the idea you are discussing.

1. Pick a rectangle. ________ Say something numerical about it.

2. Say the same thing about the similar rectangle in another group's kit.

3. Say the same thing about all Parent Rectangles, using a variable for the length of a side **of the Parent Rectangle**, and another variable for the idea you are discussing.

Appendix F. A possible strand of instruction around rectangle families.

This plan is more complete than the research strand described in the chapter. Its premise is that the students have explored polynomials only by way of the algebra tiles activities described in chapter 1, although in the research strand this activity was actually the first exposure to polynomials which the students had. As you can see in the description below, opportunities to formalize the instruction of particular factoring elements have been included, although with the research strand, only the experiential components were developed. Teachers will have to decide for themselves where to include non-contextual practice at factoring algebraic expressions.

1. --Describing the groups' Parent Rectangles, specifically and generally (whole-class).
   --Setting a variable name for the dimension, setting a strategy for changing any variable name (whole-class).
   --Describing the area of the groups' Parent Rectangles, setting a variable name for the area. Introduce concept of formula briefly (whole-class).

2. --Finding out the ant-trip and the offices with windows, get variable names, and express specifically and generally (small groups).
   --Discussing the formulas the students have, insisting on using nothing more than the variable name for dimension, the variable name for area, and specific numerical values. Define formula as a calculation procedure that describes an organized counting of a value (whole-class).

3. --Cutting out other rectangles close to the Parent Rectangle (small groups).
   --Getting all the rectangles named (whole-class).
   --Discussing the First Child $(x - 1)$ by $(x + 1)$: is the area the same as the Parent's? (small group and whole-class)

4. --Consolidating what we have: a learning sheet for the Parent Rectangle and the First Child. writing and drawing and expressing numerically the dimensions, area, anttrip and windowoffices for each (small groups).

5. --Grid mulitplying: why is $(x-1)(x+1)$ the same as x^$2 - 1$? (whole-class)
   --why is $2(x-1) + 2(x+1)$ the same as $4x$? (whole-class)
   --rules for adding and multiplying polynomials of one variable (whole-class)

6. --Exploring the other rectangles: dimensions, area, anttrip, and officewindows, generally and specifically, in words, sketches, actions, numerical expressions. finding multiple formulas for each concept, and making connections and comparisons among them. (small group and whole-class)

7. If desired, a test can be built around the question of a square of unspecified size that grows outward from all its sides. For example, a square city with streets and avenues and square blocks can be imagined to grow outward by one block per year in each direction. Students can be asked to describe how much larger it gets each year, in sketches, words, and algebraic expressions. (individual or small group)

8. --Comparing all the perfect squares, the differences of squares, and the factorible trinomials among the areas in the Rectangle Family (combinations of whole-class, small group, individual).

Appendix G. Daily homework questions and other student writing

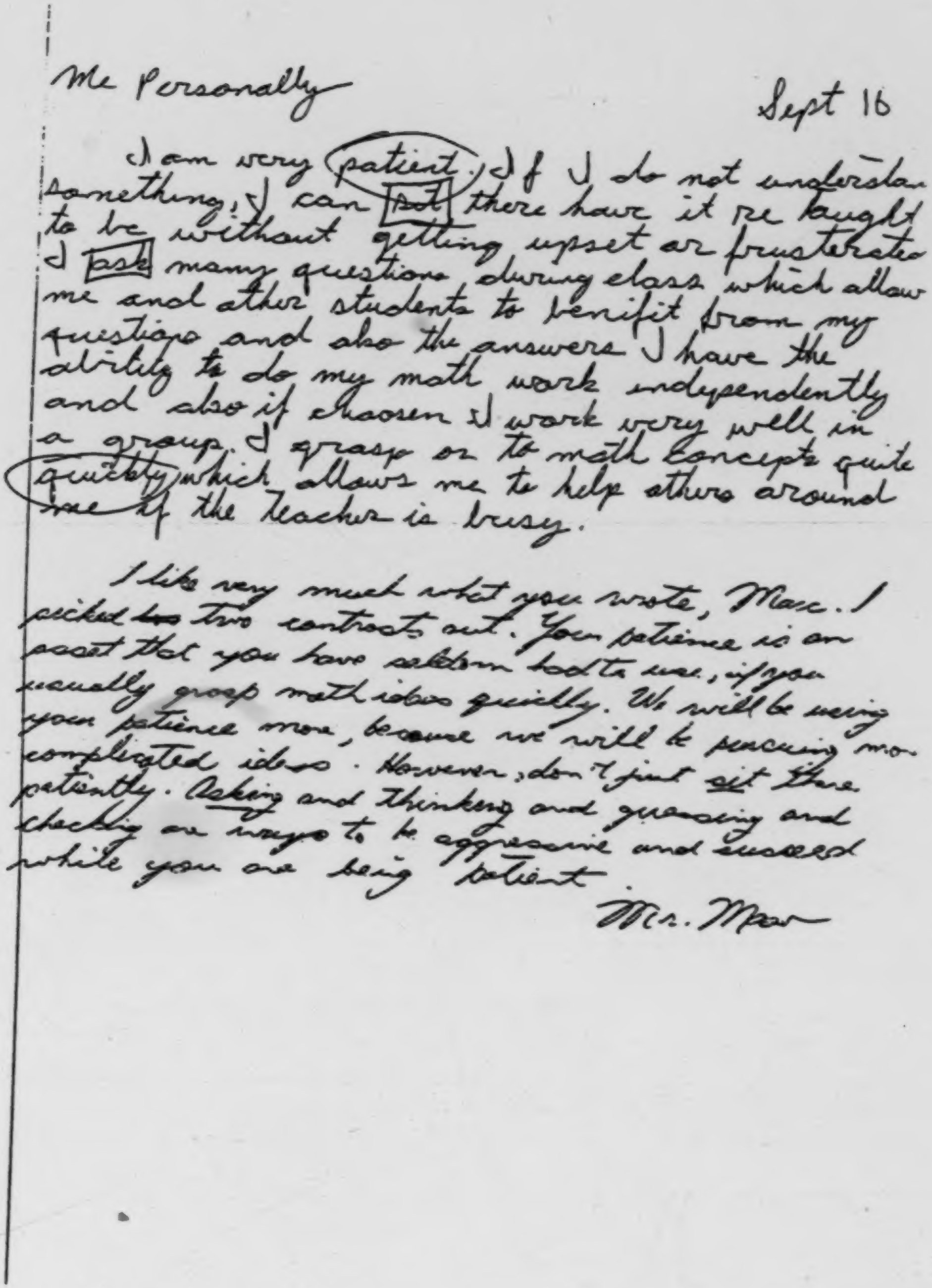
Homework questions were always said orally, and tended to be said in ways that were open for taking different meanings. These statements are short summaries of the general idea of each day's question.

September 11	Side List: What makes a good math student.
September 11	Homework: Three things that make me a good math student.
September 14	Side list: Differences about this math class.
September 18	Homework: Tell about mark worries from Mr. Mason
September 28	Homework: What good is it to you to explain math in words?
October 7	Homework: Pick out one or two ways your way of learning is different. (actions)
October 20	Homework: What mark would you give yourself for the first three areas? Why?
October 25	Parent Report (rough draft) viewed.
October 29	Homework: What do you think is coming up with the Painted Cubes?
November 5	Homework: Answer my response to your marks comment.
November 17	Homework: 1. What does the graph zzupp look like for a ____ order idea? (zero, first, second, third) 2. Does your group's mark reflect your group's ____? (effort, learning, knowledge)
November 19	Homework: Describe a learning moment from a different class today.
November xx	Graphs for the painted cube collected twice
December 1	Homework: What should we write down? Why?
December 13	Homework: What should the pre-formal algebra test look like?
January 7	Homework: Ask me a question, and prepare to respond to my answer.
January 7-9	Test/Project
January 20	In-class comment on topics for the parent report
January 22	Parent Report (rough draft)
January 26	Parent Report (final draft)
January 26	Homework: What did the presentations do for you?
January 27	Worksheet: Diagonals-and-Radicals Discovery Sheet
January 27	Homework: What have the homework questions been designed to help you with?
January 28	Homework Prepare an idea for me to respond to: marking.
February 2	Homework: What is happening to the leftovers?

February 3	Homework: Respond to any one idea from last week.
February 4	Homework: How have algebra tiles helped you?
February 5	Homework: How did Jodie, Tina, and Rhonda teach?
February 16	Homework: What should we talk about in the closure conversations?
February 17	Homework: Why should I prioritize your group?
February 23	Homework: What is algebra?
March 3	Worksheets 1-5 Algebra Kit
March 3	Test and marksheet
March 3, 4	Homework: Say something about your partner's learning. Respond to their comment.
March 11	Homework: 1. What mark should our group get out of 50 on the worksheets? Why? 2. How should you get a mark out of 100 for the test?

The following four pages show examples of homework exchanges in their original form.
The examples used are all from Martin's homework.

Appendix H. Martin's homework, September 16



Me Personally

Sept 16

I am very patient. If I do not understan something, I can sit there have it re taught to be without getting upset or frusterated I ask many questions during class which allow me and other students to benifit from my questions and also the answers I have the ability to do my math work independently and also if choosen I work very well in a group. I grasp on to math concepts quite quickly which allows me to help others around me if the teacher is busy.

I like very much what you wrote, Marc. I picked ~~[illegible]~~ two contrasts out. Your patience is an asset that you have seldom had to use, if you usually grasp math ideas quickly. We will be using your patience more, because we will be pursuing more complicated ideas. However, don't just sit there patiently. Asking and thinking and guessing and checking are ways to be aggressive and succeed while you are being patient.

Mr. Mar

Appendix H. Martin's homework, January 27

'What have the homework questions been designed to help you with?

The homework questions have help me in many ways. First I believe that some of your questions are desingned to help me think about what is coming next. These questions give me a chance to think about topic related questions will be coming up next. Some of the questions are designed so that you can find out what we are having problems with, and you can make changes or give extra help. Some questions are related towards my opion on various topics. I believe that maybe you use my opions and the way I think in your research perhaps. Again my opion (as well as others) they help you make changes in our work and how to explain it.

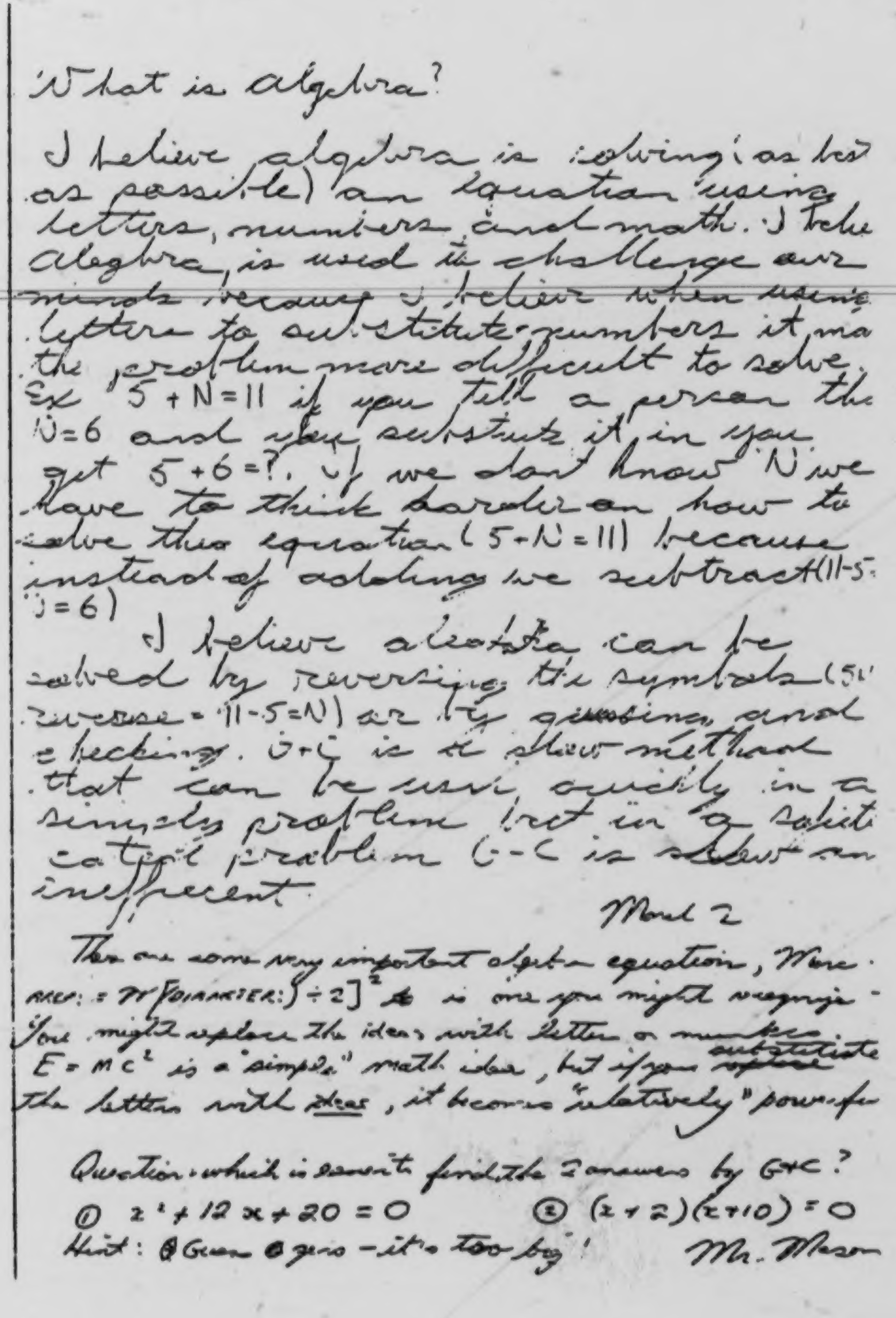
I am happy that you believe I listened to your words, Marc. You suggest I used them in my research, and to make changes in what I taught and how I taught. You suggest that I gave you extra help when homework suggested it. To be able to please you like that would make me proud of my teaching!

Your first sentence suggest that the design of the questions was so that answering would help you. Your suggestion that sometimes they helped you prepare mentally for a topic to come is correct – I'm glad you noticed!

I will only be here twice a week at most from now on. If I don't ask a question, feel free to ask one yourself and I will respond

Mr. Mason.

Appendix H. Martin's homework, February 23



What is algebra?

I believe algebra is solving (as best as possible) an equation using letters, numbers and math. I believe algebra is used to challenge our minds because I believe when using letters to substitute numbers it ma the problem more difficult to solve. Ex 5 + N = 11 if you tell a person the N=6 and you substitute it in you get 5 + 6 = ?. If we don't know N we have to think harder on how to solve this equation (5 + N = 11) because instead of adding we subtract (11-5=N=6)

I believe algebra can be solved by reversing the symbols (5' reverse = 11-5=N) or by guessing and checking. G+C is a slow method that can be used quickly in a simple problem but in a sophisticated problem G-C is slow and ineffecient.

Mod 2

There are some very important algebra equations, Mine. AREA: = π[DIAMETER:} ÷ 2]² ~~A~~ is one you might recognize. You might replace the ideas with letters or ~~numbers~~. E = MC² is a "simple" math idea, but if you ~~replace~~ substitute the letters with ideas, it becomes "relatively" powerful ✓

Question: which is easier to find the 2 answers by G+C?

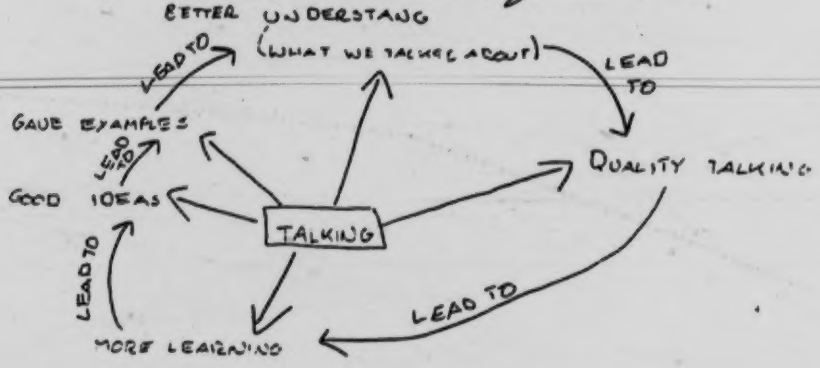
① $x^2 + 12x + 20 = 0$ ② $(x+2)(x+10) = 0$

Hint: ① Guess ② zero – it's too big! Mr. Mason

Appendix H. Martin's homework, March 4

Evaluation of Beta

What her talking lead to:


BETTER UNDERSTANG
(WHAT WE TALKED ABOUT)
LEAD TO
LEAD TO
GAVE EXAMPLES
LEAD TO
QUALITY TALKING
GOOD IDEAS
TALKING
LEAD TO
LEAD TO
MORE LEARNING

Beta has a very unique way of explaining what she did (in the aspect of learning) to achieve a solut. Beta has very good ideas which she can convert into examples, grid multiplying and drawings. She gives many examples for each solution to show her understande. She has a very good understanding of what she talked about on the test and she learned this through her work as well as her talking.

Appedix I. Patterns review sheets

Showing your learning names ____________________

____

date ______________ ____________________

Handshakes

1. The arithmetic for the number of handshakes for 80 people is

$$\frac{80 \times 79}{2}$$ Explain why this is correct.

2. A whole bunch of people, all married, each say "Hello" to everyone else in the room, except their spouses. What arithmetic will count the number of "Hellos?"

Passways

1. The formula for the number of passways includes subtracting 3, multiplying, and dividing by two. Why?

2. How many diagonals does a 20-sided polygon have?

Odd Jobs

1. What is the first odd number? What is the 500th?

2. Why does the sum of the first 20 odd numbers add up to 20 x 20?

Showing your learning names ____________________

date ____________ ____________________

____

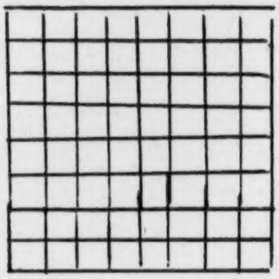
Rectangles

A. The original square:

1. Make a label for the idea of "the edge of the original rectangle." Make up a value for this for the specific example.

2. Give labels for the ideas of "how many mini-squares are inside the original square" and "how far an ant would travel around the original square". Then identify different ways those ideas could be determined, using the values and labels you chose in # 1. Put your answers in the two columns below the two sketches.

Specific example



General case

B. A new rectangle: Imagine a rectangle that is built from the original square by growing larger by three centimetres in length, and shrinking one centimetre in width. Put your work on the back of this page, organized well enough for me to mark.

1. Give labels for the ideas of "how many mini-squares are inside this rectangle" and "how far an ant would travel around the whole rectangle."

2. Then identify different ways those ideas could be determined. It is especially important to think of some ways that use the value and label you chose for the edge of the original square.

3. Do some grid arithmetic to show how some of your answers to #2 are related.

Showing your learning names ____________________

date ______________ ____________________

____

The Painted Cube

1. Make a label for the number of mini-cubes along each edge; and make up a value for the specific example. Put them on the sketches.

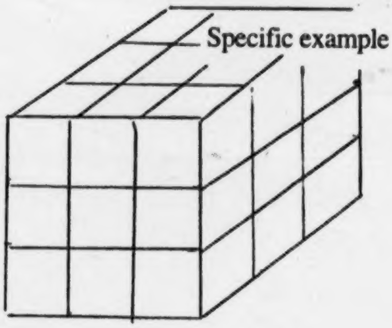
2. Give the following ideas labels:

The number of cubes with no paint at all; ______

The number of mini-squares on one face of the cube; ______

The number of mini-cubes an ant would touch, traveling all the way around, where the cube touches the table it is on. ______

3. Show ways of calculating these three ideas, using the label and the value you gave in #1.



General case

3. On the back of this page, figure out what order each of these three ideas is. Tell what the graph would look like.

Appendix J. Some patterns inquiry test questions

1. The brother-sister dance club [chosen by Colleen, Ruby, and Kelsey]

A certain dance club only lets people in, if they are in pairs, a brother and a sister. One night there were some people at the club. Six months later, two marriages occurred between people at the club that night! How many different arrangements can you make that might be a list of who married whom?

2. The league.

A certain sport is organizing into a league. How many games should there be before the playoffs start? (Make some basic rules. Then answer the question, according to the rules, depending on how many teams there are.)

3. The tables. [chosen by Lorna, Rose, and Benazhir]

The brown tables in the staff room seat two along each edge, and one on each end. If a single row of tables, touching each other either end to end or side to side, is arranged for a meeting, how many people can sit down?

4. The water containers. [chosen by Monuel, Silver, Maria, and Hilde]

The Canadian government is shipping 20 dm by 20 dm sheets of metal to Somalia. There, the edges will be folded up to make containers. How much water will they hold? (If you count each cube that is a dm by a dm by a dm, that is one litre!)

5. The Tower of Hanoi.

Some radioactive disks, all different sizes, are stacked on a special pad. They are in order of size, from the smallest on top to the largest on bottom. There are two other special pads available for holding these disks. Move the pile to one of the other pads, but follow these safety rules (or Kaboom!):

--disks can only go on one of the three pads, or on top of other disks on a pad.

--only one disk may be moving from one pad or pile to another at any time.

--never put a larger disk on a smaller disk.

How many disk-moves will moving the pile take?

[Two other groups chose to extend their inquiry of questions from the review sheets. One group chose to prepare lessons for the class on the formal terminology of polynomials in the textbook.]

Appendix K. Algebra tiles test

Grade nine test: Algebra tiles name ____________________

Part A has four questions. You have only 20 minutes. Work privately. Use any resources you wish (except live resources!). The only stuff I will mark is what you put on this page, in pencil. I will tell you about part B later.

All the questions on this test begin with this algebra expression: $x^2 + 10x + 16$.

1. Do some good algebra with this expression.

2. If you made a rectangle in question 1, you know that this expression is not PRIME. What numbers could REPLACE the 16, without making it prime?

Grade nine test: Algebra tiles page 2 of 2

3. Starting with the original expression, what numbers could replace the 10 without making it prime?

4. I suspect that at least one of your answers to questions 2 and 3 were PERFECT SQUARES. What can you tell me about perfect squares?

Appendix L. Algebra tiles test feedback sheet

Grade nine Feedback sheet: Algebra tiles test name Maria

relations

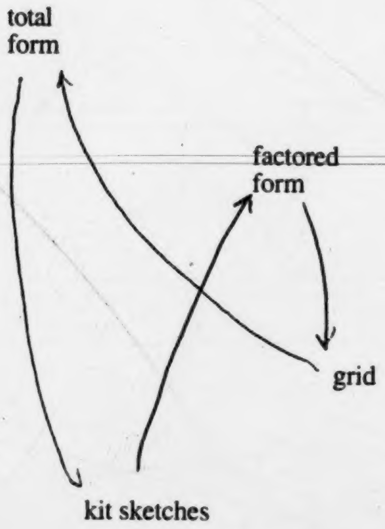
graphs

patterns & rules

terminology

multiplying

other arithmetic



Question 2	**Question 3**	**Question 4**
Positive only	Positive only	Positive only
--one answer ✓	--one answer ✓	--one answer ✓
--another ✓	--another ✓	--another
--many ✓	--many ✓	--many
--all ✓	--all	--all
Negative too	Negative too	Negative too + equal sides
--consider	--consider	--consider
--one	--one	--one
--another	--another	--another + symmetry
--many	--many	--many
--all	--all	--all
Patterns + composite	Patterns	Patterns + square root
--notice one ✓	--notice one ✓	--notice one ✓
--express one ✓	--express one ✓	--express one ✓
--explain one	--explain one	--explain one
--notice another	--notice another	--notice another
--express another	--express another	--express another
--explain another	--explain another	--explain another

Grade nine feedback sheet: Algebra tiles test

page 2 of 2

A system for the test:

After reading everyone's advice on how to determine a mark from the test, I was left with two strong concepts. One, adding up the points from the various details (two points per element and two points per connection, for example) doesn't say what matters. Second, students want to be treated as people by a person, not treated as objects by a marking system. Despite this last factor, I feel obliged to be systematic about scoring the test. I found five things that mattered to your success in algebra, for which the test could provide evidence.

A. basic skills cycle: from total form to factored form to total form. Great would be worth 50.

B. learning processes: expressing and listening interactively, wondering, putting thoughts together into new parts (Part B particularly). Great would be worth 20.

C. searching for polynomials: exploring for possibilities, especially systematically, considering a variety, hunting for reasons (Q2 and 3). Great would be worth 10.

D. patterns searched for, some simple patterns found and explained. Great would be worth 10.

E. other (algebra perfect squares, extra components in cycle, explaining, ...). Great would be worth 10.

Silver: $50 + 20 + 8 + 20 + 20 > 100$
Monuel: $50 + 10 + 12 + 5 + 10 = 87$
Maria: $50 + 15 + 12 + 5 + 12 = 94$
Hilde: $50 + 15 + 10 + 20 + 5 = 100$

Wow! What a set of scores! The big difference among you is that two of you noticed and described some patterns. I think the other two may have noticed them, but perhaps you did not think of wondering as being part of a test! The perfect squares question was also well done -- I think Hilde chose to chase her patterns more instead of explore this question. The other difference is in Part B of the test with a new partner. Monuel, you didn't make the magic happen the way I so often hear it when I listen to your group. Being able to make magic with a new partner IS tough to do.